

Straight Lines

INTRODUCTION

A systematic study of geometry by using algebra was first carried out by French mathematician *R Descartes* in 1637. This subject is now known as *Coordinate Geometry*.

In earlier classes, we have studied about coordinate axis, coordinate plane, points in a plane, distance between two points and section formulae.

Here, we give, in brief, the results derived so far.

1. Distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

2. Area of $\triangle ABC$ whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$\Delta = \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \text{ sq units.}$$

REMARK The points A, B, C are collinear $\Leftrightarrow$ area of $\triangle ABC = 0$.

3. (i) If the point $P(x, y)$ divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$ then

$$x = \frac{mx_2 + nx_1}{m+n} \quad \text{and} \quad y = \frac{my_2 + ny_1}{m+n}.$$

- (ii) If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ then

$$x = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \frac{1}{2}(y_1 + y_2).$$

SOLVED EXAMPLES

EXAMPLE 1 Find the distance between the points $(2, -3)$ and $(-6, 3)$.

SOLUTION Let $A(2, -3)$ and $B(-6, 3)$ be the given points. Then,

$$\begin{aligned} AB &= \sqrt{(-6-2)^2 + \{3-(-3)\}^2} = \sqrt{(-8)^2 + (3+3)^2} = \sqrt{(-8)^2 + 6^2} \\ &= \sqrt{64 + 36} = \sqrt{100} = 10 \text{ units.} \end{aligned}$$



EXAMPLE 2 Using the distance formula, prove that the points $A(-2, 3)$, $B(1, 2)$ and $C(7, 0)$ are collinear.

SOLUTION We have

$$AB = \sqrt{(1 + 2)^2 + (2 - 3)^2} = \sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ units};$$

$$BC = \sqrt{(7 - 1)^2 + (0 - 2)^2} = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10} \text{ units};$$

$$AC = \sqrt{(7 + 2)^2 + (0 - 3)^2} = \sqrt{9^2 + (-3)^2} = \sqrt{90} = 3\sqrt{10} \text{ units}.$$

$$\therefore AB + BC = (\sqrt{10} + 2\sqrt{10}) \text{ units} = 3\sqrt{10} \text{ units} = AC.$$

Thus, $AB + BC = AC$, showing that the points A, B, C are collinear.

SOME IMPORTANT RESULTS

A quadrilateral is a

- (i) rectangle, if its opposite sides are equal and the diagonals are equal;
- (ii) square, if its all sides are equal and the diagonals are equal;
- (iii) a parallelogram but not a rectangle, if its opposite sides are equal but the diagonals are not equal;
- (iv) a rhombus but not a square, if its all sides are equal but the diagonals are not equal.

EXAMPLE 3 Prove that the points $(0, 5)$, $(-2, -2)$, $(5, 0)$ and $(7, 7)$ are the vertices of a rhombus.

SOLUTION Let the given points be $A(0, 5)$, $B(-2, -2)$, $C(5, 0)$ and $D(7, 7)$. Then, $ABCD$ is a quadrilateral in which:

$$AB = \sqrt{(-2 - 0)^2 + (-2 - 5)^2} = \sqrt{(-2)^2 + (-7)^2} = \sqrt{4 + 49} = \sqrt{53} \text{ units};$$

$$BC = \sqrt{(5 + 2)^2 + (0 + 2)^2} = \sqrt{7^2 + 2^2} = \sqrt{49 + 4} = \sqrt{53} \text{ units};$$

$$CD = \sqrt{(7 - 5)^2 + (7 - 0)^2} = \sqrt{2^2 + 7^2} = \sqrt{4 + 49} = \sqrt{53} \text{ units};$$

$$DA = \sqrt{(7 - 0)^2 + (7 - 5)^2} = \sqrt{7^2 + 2^2} = \sqrt{49 + 4} = \sqrt{53} \text{ units}.$$

$$\text{Thus, } AB = BC = CD = DA = \sqrt{53} \text{ units}.$$

$$\text{Also, } AC = \sqrt{(5 - 0)^2 + (0 - 5)^2} = \sqrt{5^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2} \text{ units}$$

$$\begin{aligned} \text{And, } BD &= \sqrt{(7 + 2)^2 + (7 + 2)^2} = \sqrt{9^2 + 9^2} = \sqrt{81 + 81} \\ &= \sqrt{162} = 9\sqrt{2} \text{ units.} \end{aligned}$$

$$\therefore AC \neq BD.$$

Thus, $ABCD$ is a quadrilateral all of whose sides are equal but its diagonals are not equal.

Hence, $ABCD$ is a rhombus.

EXAMPLE 4 Find the area of the triangle whose vertices are $A(4, 4)$, $B(3, -16)$ and $C(3, -2)$.

SOLUTION Let $(x_1 = 4, y_1 = 4)$; $(x_2 = 3, y_2 = -16)$ and $(x_3 = 3, y_3 = -2)$.



Then, area of $\triangle ABC$ is given by

$$\begin{aligned}\Delta &= \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= \frac{1}{2} \cdot |4(-16 + 2) + 3(-2 - 4) + 3(4 + 16)| \\ &= \frac{1}{2} \cdot |-56 - 18 + 60| = \frac{1}{2} \cdot |-14| = \left(\frac{1}{2} \times 14\right) = 7 \text{ sq units.}\end{aligned}$$

Hence, the area of the given triangle is 7 sq units.

EXAMPLE 5 Find the coordinates of the point which divides the line segment joining the points $A(5, -2)$ and $B(9, 6)$ in the ratio 3 : 1.

SOLUTION Here $(x_1 = 5, y_1 = -2)$ and $(x_2 = 9, y_2 = 6)$. Also, $m = 3$ and $n = 1$.

Let the required point be $P(x, y)$. Then,

$$\begin{aligned}x &= \frac{(mx_2 + nx_1)}{(m + n)} = \frac{(3 \times 9) + (1 \times 5)}{(3 + 1)} = \frac{32}{4} = 8, \\ y &= \frac{(my_2 + ny_1)}{(m + n)} = \frac{(3 \times 6) + \{1 \times (-2)\}}{(3 + 1)} = \frac{(18 - 2)}{4} = \frac{16}{4} = 4.\end{aligned}$$

Hence, the required point is $(8, 4)$.

EXAMPLE 6 Find the coordinates of the midpoint of the line segment joining the points $A(-2, -5)$ and $B(3, -1)$.

SOLUTION Let the required point be $P(x, y)$. Then,

$$x = \frac{(-2 + 3)}{2} = \frac{1}{2}, y = \frac{-5 + (-1)}{2} = -3.$$

Hence, the required point is $\left(\frac{1}{2}, -3\right)$.

EXAMPLE 7 In what ratio is the line joining $A(-1, 1)$ and $B(5, 7)$ divided by the line $x + y = 4$?

SOLUTION Let the given line divide AB in the ratio $m : 1$.

Then, the point of division is $C\left(\frac{5m - 1}{m + 1}, \frac{7m + 1}{m + 1}\right)$.

This point C must lie on the line $x + y = 4$.

$$\begin{aligned}\therefore \frac{5m - 1}{m + 1} + \frac{7m + 1}{m + 1} &= 4 \Leftrightarrow (5m - 1) + (7m + 1) = 4(m + 1) \\ &\Leftrightarrow 8m = 4 \Leftrightarrow m = \frac{1}{2}.\end{aligned}$$

So, the required ratio is $\frac{1}{2} : 1$, i.e., $1 : 2$.



EXERCISE 20A

1. Find the distance between the points:
(i) $A(2, -3)$ and $B(-6, 3)$ (ii) $C(-1, -1)$ and $D(8, 11)$
(iii) $P(-8, -3)$ and $Q(-2, -5)$ (iv) $R(a + b, a - b)$ and $S(a - b, a + b)$
2. Find the distance of the point $P(6, -6)$ from the origin.
3. If a point $P(x, y)$ is equidistant from the points $A(6, -1)$ and $B(2, 3)$, find the relation between x and y .
4. Find a point on the x -axis which is equidistant from the points $A(7, 6)$ and $B(-3, 4)$.
5. Find the distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$, when (i) AB is parallel to the x -axis (ii) AB is parallel to the y -axis.
6. A is a point on the x -axis with abscissa -8 and B is a point on the y -axis with ordinate 15 . Find the distance AB .
7. Find a point on the y -axis which is equidistant from $A(-4, 3)$ and $B(5, 2)$.
8. Using the distance formula, show that the points $A(3, -2)$, $B(5, 2)$ and $C(8, 8)$ are collinear.
9. Show that the points $A(7, 10)$, $B(-2, 5)$ and $C(3, -4)$ are the vertices of an isosceles right-angled triangle.
10. Show that the points $A(1, 1)$, $B(-1, -1)$ and $C(-\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle each of whose sides is $2\sqrt{2}$ units.
11. Show that the points $A(2, -2)$, $B(8, 4)$, $C(5, 7)$ and $D(-1, 1)$ are the angular points of a rectangle.
12. Show that $A(3, 2)$, $B(0, 5)$, $C(-3, 2)$ and $D(0, -1)$ are the vertices of a square.
13. Show that $A(1, -2)$, $B(3, 6)$, $C(5, 10)$ and $D(3, 2)$ are the vertices of a parallelogram.
14. Show that the points $A(2, -1)$, $B(3, 4)$, $C(-2, 3)$ and $D(-3, -2)$ are the vertices of a rhombus.
15. If the points $A(-2, -1)$, $B(1, 0)$, $C(x, 3)$ and $D(1, y)$ are the vertices of a parallelogram, find the values of x and y .
16. Find the area of $\triangle ABC$ whose vertices are $A(-3, -5)$, $B(5, 2)$ and $C(-9, -3)$.
17. Show that the points $A(-5, 1)$, $B(5, 5)$ and $C(10, 7)$ are collinear.
18. Find the value of k for which the points $A(-2, 3)$, $B(1, 2)$ and $C(k, 0)$ are collinear.
19. Find the area of the quadrilateral whose vertices are $A(-4, 5)$, $B(0, 7)$, $C(5, -5)$ and $D(-4, -2)$.
20. Find the area of $\triangle ABC$, the midpoints of whose sides AB , BC and CA are $D(3, -1)$, $E(5, 3)$ and $F(1, -3)$ respectively.
21. Find the coordinates of the point which divides the join of $A(-5, 11)$ and $B(4, -7)$ in the ratio $2 : 7$.



22. Find the ratio in which the x -axis cuts the join of the points $A(4, 5)$ and $B(-10, -2)$. Also, find the point of intersection.
23. In what ratio is the line segment joining the points $A(-4, 2)$ and $B(8, 3)$ divided by the y -axis? Also, find the point of intersection.

ANSWERS (EXERCISE 20A)

1. (i) 10 units (ii) 15 units (iii) $2\sqrt{10}$ units (iv) $(2\sqrt{2}b)$ units 2. $6\sqrt{2}$ units
3. $x - y = 3$ 4. $P(3, 0)$ 5. (i) $|x_2 - x_1|$ (ii) $|y_2 - y_1|$
6. 17 units 7. $P(0, -2)$ 15. $x = 4, y = 2$ 16. 29 sq units
18. $k = 7$ 19. 60.5 sq units 20. 8 sq units 21. $P(-3, 7)$
22. $(5 : 2), P(-6, 0)$ 23. $(1 : 2), P\left(0, \frac{7}{3}\right)$

HINTS TO SOME SELECTED QUESTIONS

4. Let the required point be $P(x, 0)$.
Then, $AP = BP \Leftrightarrow AP^2 = BP^2$.
5. (i) When AB is parallel to the x -axis, then A and B have the same ordinate.
Then, $A(x_1, y)$ and $B(x_2, y)$ are the given points. So, $AB = |x_2 - x_1|$.
- (ii) When AB is parallel to the y -axis, then A and B have the same abscissa.
Then, $A(x, y_1)$ and $B(x, y_2)$ are the given points. So, $AB = |y_2 - y_1|$.
6. Given points are $A(-8, 0)$ and $B(0, 15)$.
7. Let the required point be $P(0, y)$.
8. Show that $AB + BC = AC$.
11. Show that $AB = DC, BC = AD$ and diag. $AC = \text{diag. } BD$.
12. Show that $AB = BC = CD = DA$ and diag. $AC = \text{diag. } BD$.
13. Show that $AB = CD$ and $BC = AD$.
15. Midpoint of $AC = \text{Midpoint of } BD$.
17. Show that area of $\triangle ABC$ is zero.
19. Area of quad. $ABCD = \text{area}(\triangle ABC) + \text{area}(\triangle ACD)$.
20. Let $\triangle ABC$ be given and let D, E, F be the midpoints of AB, BC and CA respectively.
Then, $\text{ar}(\triangle ABC) = 4 \times \text{ar}(\triangle DEF)$.
22. Let the required ratio be $k : 1$ and let $P(x, 0)$ be the point of division. Then,
$$\frac{k \times (-2) + 1 \times 5}{k + 1} = 0 \Leftrightarrow k = \frac{5}{2}$$

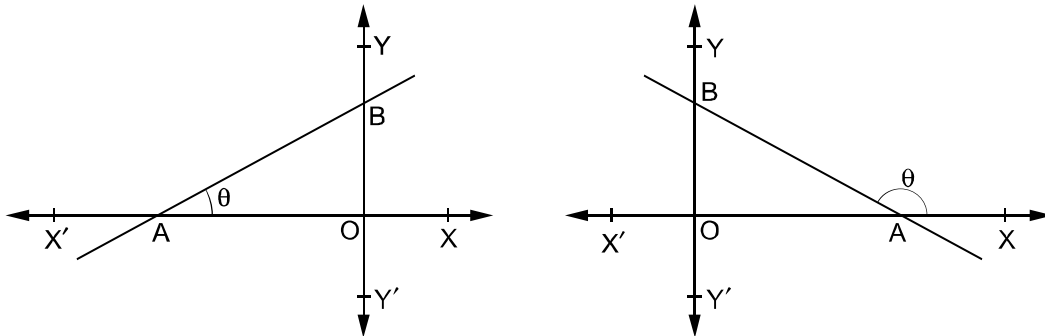
So, the required ratio = $5 : 2$.
23. Let the ratio be $k : 1$ and let $P(0, y)$ be the point of division.



SLOPE OF A LINE

INCLINATION OF A LINE

The angle of inclination or simply the inclination of a line is the angle θ which the part of the line above the x -axis makes with the positive direction of the x -axis, measured in anticlockwise direction.



Clearly, $0^\circ \leq \theta < 180^\circ$.

- REMARKS (i) The inclination of a line parallel to the x -axis or the x -axis itself is 0° .
(ii) The inclination of a line parallel to the y -axis or the y -axis itself is 90° .

HORIZONTAL LINE Any line parallel to the x -axis or the x -axis itself is called a horizontal line.

VERTICAL LINE Any line parallel to the y -axis or the y -axis itself is called a vertical line.

OBLIQUE LINE A line which is neither horizontal nor vertical is called an oblique line.

SLOPE OR GRADIENT OF A LINE

If θ is the inclination of a non-vertical line, then $m = \tan \theta$ is called the slope of the line.

EXAMPLE 1 Find the slope of a line whose inclination is

- (i) 45° (ii) 60° (iii) 150°

SOLUTION Let m be the slope of the line. Then,

- (i) $m = \tan 45^\circ = 1$.
(ii) $m = \tan 60^\circ = \sqrt{3}$.
(iii) $m = \tan 150^\circ = \tan (180^\circ - 30^\circ) = -\tan 30^\circ = \frac{-1}{\sqrt{3}}$.

REMARK 1 The slope of a horizontal line is 0.

We know that the inclination of a horizontal line is 0° .

So, slope of a horizontal line is $m = \tan 0^\circ = 0$.

REMARK 2 The slope of a vertical line is not defined.

We know that the inclination of a vertical line is 90° .

So, slope of a vertical line is $m = \tan 90^\circ$, which is not defined.



EXAMPLE 2 What is the inclination of a line whose slope is

- (i) zero? (ii) positive? (iii) negative? (iv) not defined?

SOLUTION Let θ be the inclination of the given line. Then, $m = \tan \theta$.

$$\begin{aligned} \text{(i) } m = 0 &\Rightarrow \tan \theta = 0 \\ &\Rightarrow \theta = 0^\circ \end{aligned} \quad [\because 0^\circ \leq \theta < 180^\circ]$$

$$\begin{aligned} \text{(ii) } m > 0 &\Rightarrow \tan \theta > 0 \\ &\Rightarrow \theta \text{ lies between } 0^\circ \text{ and } 90^\circ \\ &\Rightarrow \theta \text{ is acute.} \end{aligned}$$

$$\begin{aligned} \text{(iii) } m < 0 &\Rightarrow \tan \theta < 0 \\ &\Rightarrow \theta \text{ lies between } 90^\circ \text{ and } 180^\circ \\ &\Rightarrow \theta \text{ is obtuse.} \end{aligned}$$

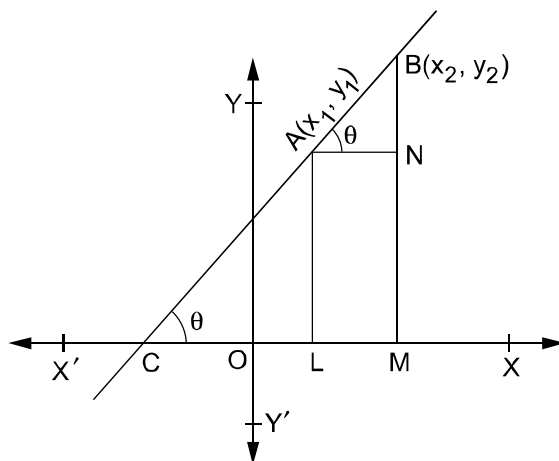
(iv) We know that a vertical line is the only line whose slope is not defined. And, the inclination of a vertical line is 90° .

Hence, the inclination of a line whose slope is not defined, is 90° .

SLOPE OF A LINE PASSING THROUGH TWO GIVEN POINTS

THEOREM 1 Prove that the slope of a non-vertical line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

PROOF Let $X'OX$ and YOY' be the coordinate axes. Let us consider a non-vertical line CAB , passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let θ be its inclination.



Draw $AL \perp OX$, $BM \perp OX$ and $AN \perp BM$.

Then, $\angle BAN = \angle ACX = \theta$ (corr. $\angle$).

From right triangle ANB , we have:

$$\tan \theta = \frac{BN}{AN} = \frac{(BM - NM)}{LM} = \frac{(BM - AL)}{(OM - OL)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$\Rightarrow m = \frac{(y_2 - y_1)}{(x_2 - x_1)} \quad [\because \tan \theta = m].$$



Hence, the slope of the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by, $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

EXAMPLE 3 Find the slope of the line passing through the points

(i) $(-2, 3)$ and $(8, -5)$ (ii) $(4, -3)$ and $(6, -3)$ (iii) $(3, -1)$ and $(3, 2)$

SOLUTION (i) Let $A(-2, 3)$ and $B(8, -5)$ be the given points. Then,

$$\text{slope of } AB = \frac{-5 - 3}{8 - (-2)} = \frac{-8}{10} = \frac{-4}{5} \quad \left[\because m = \frac{(y_2 - y_1)}{(x_2 - x_1)} \right]$$

(ii) Let $C(4, -3)$ and $D(6, -3)$ be the given points. Then,

$$\text{slope of } CD = \frac{-3 - (-3)}{6 - 4} = \frac{-3 + 3}{2} = \frac{0}{2} = 0.$$

ALITER The points $C(4, -3)$ and $D(6, -3)$ have the same y -coordinate. So, CD is a line parallel to the x -axis. Hence, its slope is 0.

(iii) Let $P(3, -1)$ and $Q(3, 2)$ be the given points. Then,

$$\text{slope of } PQ = \frac{2 - (-1)}{3 - 3} = \frac{3}{0}, \text{ which is not defined.}$$

ALITER The points $P(3, -1)$ and $Q(3, 2)$ have the same x -coordinate. So, PQ is a vertical line.

Hence, its slope is not defined.

EXAMPLE 4 If the slope of the line passing through the points $(2, 5)$ and $(x, 3)$ is 2, find the value of x .

SOLUTION Let $A(2, 5)$ and $B(x, 3)$ be the given points. Then,

$$\text{slope of } AB = \frac{3 - 5}{x - 2} = \frac{-2}{(x - 2)}.$$

$$\begin{aligned} \therefore \frac{-2}{(x - 2)} = 2 &\Leftrightarrow 2x - 4 = -2 \\ &\Leftrightarrow 2x = 2 \Leftrightarrow x = 1. \end{aligned}$$

Hence, $x = 1$.

EXAMPLE 5 Find the value of x so that the inclination of the line joining the points $(x, -3)$ and $(2, 5)$ is 135° .

SOLUTION Let $A(x, -3)$ and $B(2, 5)$ be the given points. Then,

$$\text{slope of } AB = \frac{5 - (-3)}{2 - x} = \frac{8}{(2 - x)}.$$

But, slope of $AB = \tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1$.

$$\therefore \frac{8}{(2 - x)} = -1 \Leftrightarrow x - 2 = 8 \Leftrightarrow x = 10.$$

Hence, $x = 10$.



EXAMPLE 6 Find the angle between the x-axis and the line joining the points (3, -1) and (4, -2).

SOLUTION Let A(3, -1) and B(4, -2) be the given points and let m be the slope of the line AB. Then,

$$m = \frac{-2 - (-1)}{4 - 3} = \frac{-2 + 1}{1} = -1. \quad \left[\because m = \frac{(y_2 - y_1)}{(x_2 - x_1)} \right]$$

Let θ be the angle between the x-axis and the line AB. Then,

$$\tan \theta = m = -1 = -\tan 45^\circ = \tan (180^\circ - 45^\circ) = \tan 135^\circ.$$

$$\therefore \theta = 135^\circ.$$

Hence, the required angle is 135° .

SLOPES OF PARALLEL LINES

THEOREM 2 Prove that two non-vertical lines are parallel, if and only if their slopes are equal.

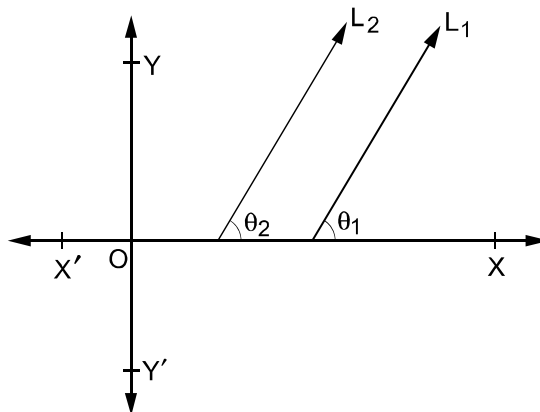
PROOF Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively.

First we assume that L_1 and L_2 are parallel.

Then, L_1 and L_2 are parallel $\Rightarrow \theta_1 = \theta_2$

$$\Rightarrow \tan \theta_1 = \tan \theta_2$$

$$\Rightarrow m_1 = m_2.$$



Thus, non-vertical parallel lines have equal slopes.

Conversely, let $m_1 = m_2$. Then,

$$m_1 = m_2 \Rightarrow \tan \theta_1 = \tan \theta_2$$

$$\Rightarrow \theta_1 = \theta_2 \quad [\because 0^\circ \leq \theta < 180^\circ]$$

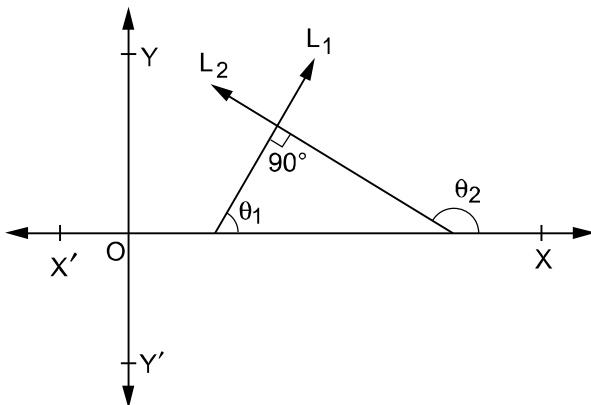
$$\Rightarrow L_1 \text{ and } L_2 \text{ are parallel.}$$

Thus, non-vertical lines having equal slopes, are parallel. Hence, two non-vertical lines are parallel if and only if their slopes are equal.

SLOPES OF PERPENDICULAR LINES

THEOREM 3 Prove that two non-vertical lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

PROOF Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively. Then, $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$.



First we assume that L_1 and L_2 are perpendicular to each other. Then,

$$\theta_2 = (90^\circ + \theta_1)$$

$$\Rightarrow \tan \theta_2 = \tan (90^\circ + \theta_1) = -\cot \theta_1$$

$$\Rightarrow \tan \theta_2 = -\frac{1}{\tan \theta_1}$$

$$\Rightarrow m_2 = -\frac{1}{m_1} \Rightarrow m_1 m_2 = -1.$$

Thus, when $L_1 \perp L_2$, then $m_1 m_2 = -1$.

Conversely, let L_1 and L_2 be two lines with slopes m_1 and m_2 respectively such that $m_1 m_2 = -1$.

Let $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$. Then,

$$m_1 m_2 = -1 \Rightarrow \tan \theta_1 \cdot \tan \theta_2 = -1$$

$$\Rightarrow \tan \theta_2 = \frac{-1}{\tan \theta_1} = -\cot \theta_1 = \tan (90^\circ + \theta_1)$$

$$\Rightarrow \theta_2 = 90^\circ + \theta_1 \Rightarrow \theta_2 \text{ and } \theta_1 \text{ differ by } 90^\circ$$

$$\Rightarrow L_1 \text{ and } L_2 \text{ are perpendicular to each other.}$$

Thus, when $m_1 m_2 = -1$, then $L_1 \perp L_2$.

Hence, L_1 and L_2 are perpendicular to each other, if and only if $m_1 m_2 = -1$.

SUMMARY

Let L_1 and L_2 be two lines whose slopes are m_1 and m_2 respectively. Then,

$$(i) L_1 \parallel L_2 \Leftrightarrow m_1 = m_2$$

$$(ii) L_1 \perp L_2 \Rightarrow m_1 m_2 = -1$$



EXAMPLE 7 Show that the line joining the points $(2, -3)$ and $(-5, 1)$ is parallel to the line joining the points $(7, -1)$ and $(0, 3)$.

SOLUTION Let $A(2, -3)$, $B(-5, 1)$, $C(7, -1)$ and $D(0, 3)$ be the given points.

$$\text{Then, slope of } AB = \frac{1 - (-3)}{-5 - 2} = \frac{4}{-7} = -\frac{4}{7}$$

$$\text{And, slope of } CD = \frac{3 - (-1)}{0 - 7} = \frac{4}{-7} = -\frac{4}{7}.$$

$\therefore$ slope of AB = slope of CD .

Hence, $AB \parallel CD$.

EXAMPLE 8 Show that the line joining the points $(2, -5)$ and $(-2, 5)$ is perpendicular to the line joining the points $(6, 3)$ and $(1, 1)$.

SOLUTION Let $A(2, -5)$, $B(-2, 5)$, $C(6, 3)$ and $D(1, 1)$ be the given points.

Let m_1 and m_2 be the slopes of AB and CD respectively. Then,

$$m_1 = \text{slope of } AB = \frac{5 - (-5)}{-2 - 2} = \frac{10}{-4} = -\frac{5}{2}.$$

$$m_2 = \text{slope of } CD = \frac{1 - 3}{1 - 6} = \frac{-2}{-5} = \frac{2}{5}.$$

$$\therefore m_1 m_2 = \left(-\frac{5}{2}\right) \times \frac{2}{5} = -1.$$

Hence, $AB \perp CD$.

EXAMPLE 9 If the line through the points $(-2, 6)$ and $(4, 8)$ is perpendicular to the line through the points $(8, 12)$ and $(x, 24)$, find the value of x .

SOLUTION Let $A(-2, 6)$, $B(4, 8)$, $C(8, 12)$ and $D(x, 24)$ be the given points.

Let m_1 and m_2 be the slopes of AB and CD respectively. Then,

$$m_1 = \text{slope of } AB = \frac{(8 - 6)}{4 - (-2)} = \frac{2}{6} = \frac{1}{3}.$$

$$m_2 = \text{slope of } CD = \frac{(24 - 12)}{(x - 8)} = \frac{12}{(x - 8)}.$$

$$\text{Now, } AB \perp CD \Leftrightarrow m_1 m_2 = -1$$

$$\Leftrightarrow \frac{1}{3} \times \frac{12}{(x - 8)} = -1$$

$$\Leftrightarrow -x + 8 = 4 \Leftrightarrow x = 4.$$

Hence, the required value of x is 4.

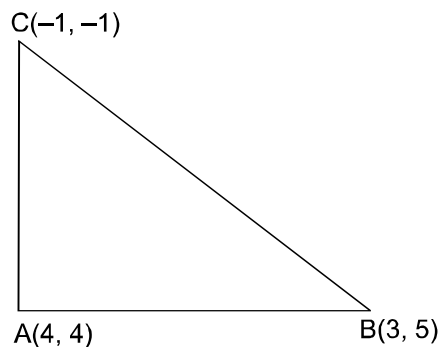
EXAMPLE 10 Without using Pythagoras's theorem show that the points $(4, 4)$, $(3, 5)$ and $(-1, -1)$ are the vertices of a right-angled triangle.

SOLUTION Let $A(4, 4)$, $B(3, 5)$ and $C(-1, -1)$ be the vertices of $\triangle ABC$.

Let m_1 and m_2 be the slopes of AB and AC respectively. Then,

$$m_1 = \text{slope of } AB = \frac{(5 - 4)}{(3 - 4)} = -1.$$





$$m_2 = \text{slope of } AC = \frac{(-1 - 4)}{(-1 - 4)} = \frac{-5}{-5} = 1.$$

$$\therefore m_1 m_2 = -1.$$

So, $AB \perp AC$ and therefore, $\angle CAB = 90^\circ$.

Hence, the given points are the vertices of a right triangle.

EXAMPLE 11 Using slopes, show that the points $(5, 1)$, $(1, -1)$ and $(11, 4)$ are collinear.

SOLUTION Let $A(5, 1)$, $B(1, -1)$ and $C(11, 4)$ be the given points. Then,

$$\text{slope of } AB = \frac{(-1 - 1)}{(1 - 5)} = \frac{-2}{-4} = \frac{1}{2}$$

$$\text{and slope of } BC = \frac{4 - (-1)}{11 - 1} = \frac{5}{10} = \frac{1}{2}.$$

$$\therefore \text{slope of } AB = \text{slope of } BC$$

$\Rightarrow AB \parallel BC$ and have a point B in common

$\Rightarrow A, B, C$ are collinear.

Hence, the given points are collinear.

EXAMPLE 12 Find the value of x for which the points $(x, -1)$, $(2, 1)$ and $(4, 5)$ are collinear.

SOLUTION Let $A(x, -1)$, $B(2, 1)$ and $C(4, 5)$ be the given collinear points.

Then, by collinearity of A, B, C we have: slope of AB = slope of BC

$$\frac{1 - (-1)}{2 - x} = \frac{(5 - 1)}{(4 - 2)} \Rightarrow \frac{2}{2 - x} = 2 \Rightarrow 2 - x = 1 \Rightarrow x = 1.$$

Hence, $x = 1$.

EXAMPLE 13 If the points $(h, 0)$, (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

SOLUTION Let $A(h, 0)$, $B(a, b)$ and $C(0, k)$ be the given collinear points.

Since the given points A, B, C are collinear, we have

$$\text{slope of } AB = \text{slope of } BC$$

$$\therefore \frac{b - 0}{a - h} = \frac{k - b}{0 - a} \Leftrightarrow \frac{b}{(a - h)} = \frac{(b - k)}{a}$$

$$\Leftrightarrow ab = (a - h)(b - k)$$



$$\Leftrightarrow ak + hb = hk$$

$$\Leftrightarrow \frac{a}{h} + \frac{b}{k} = 1 \quad [\text{on dividing both sides by } hk]$$

$$\text{Hence, } \frac{a}{h} + \frac{b}{k} = 1.$$

EXAMPLE 14 Using slopes, show that the vertices $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ are the vertices of a parallelogram.

SOLUTION Let $A(-2, -1)$, $B(4, 0)$, $C(3, 3)$ and $D(-3, 2)$ be the vertices of the given quadrilateral $ABCD$. Then,

$$\text{slope of } AB = \frac{0 - (-1)}{4 - (-2)} = \frac{1}{6};$$

$$\text{slope of } DC = \frac{3 - 2}{3 - (-3)} = \frac{1}{6};$$

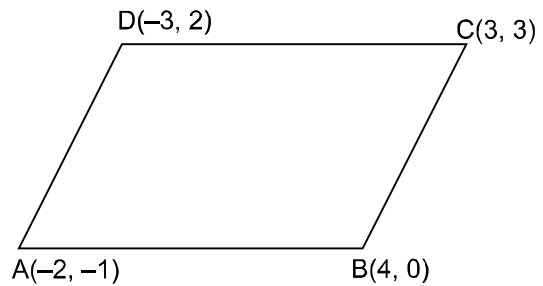
$$\text{slope of } BC = \frac{3 - 0}{3 - 4} = \frac{3}{-1} = -3;$$

$$\text{slope of } AD = \frac{2 - (-1)}{-3 - (-2)} = \frac{3}{-1} = -3;$$

$$\therefore \text{ slope of } AB = \text{slope of } DC \Rightarrow AB \parallel DC;$$

$$\text{slope of } BC = \text{slope of } AD \Rightarrow BC \parallel AD.$$

Hence, $ABCD$ is a parallelogram.



EXAMPLE 15 $A(-4, 2)$, $B(2, 6)$, $C(8, 5)$ and $D(9, -7)$ are the vertices of a quadrilateral $ABCD$. If P , Q , R , S are the midpoints of AB , BC , CD and DA respectively, using slopes, show that $PQRS$ is a parallelogram.

SOLUTION Clearly, the points P , Q , R , S are given by

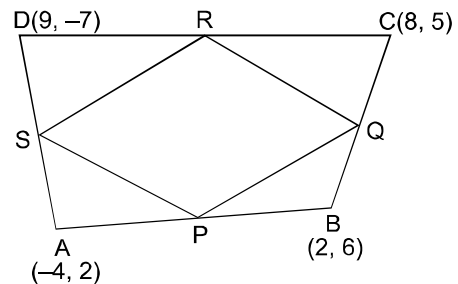
$$P(-1, 4), Q\left(5, \frac{11}{2}\right), R\left(\frac{17}{2}, -1\right) \text{ and } S\left(\frac{5}{2}, \frac{-5}{2}\right).$$

$$\therefore \text{ slope of } PQ = \frac{\left(\frac{11}{2} - 4\right)}{(5 + 1)} = \frac{3}{12} = \frac{1}{4};$$

$$\text{slope of } SR = \frac{\left(-1 + \frac{5}{2}\right)}{\left(\frac{17}{2} - \frac{5}{2}\right)} = \frac{3}{12} = \frac{1}{4};$$

$$\text{slope of } QR = \frac{\left(-1 - \frac{11}{2}\right)}{\left(\frac{17}{2} - 5\right)} = \left(\frac{-13}{2} \times \frac{2}{7}\right) = \frac{-13}{7};$$

$$\text{slope of } PS = \frac{\left(\frac{5}{2} - 4\right)}{\left(\frac{5}{2} + 1\right)} = \left(\frac{-13}{2} \times \frac{2}{7}\right) = \frac{-13}{7}.$$



$\therefore$ slope of PQ = slope of $SR \Rightarrow PQ \parallel SR$;

slope of QR = slope of $PS \Rightarrow QR \parallel PS$.

Hence, $PQRS$ is a parallelogram.

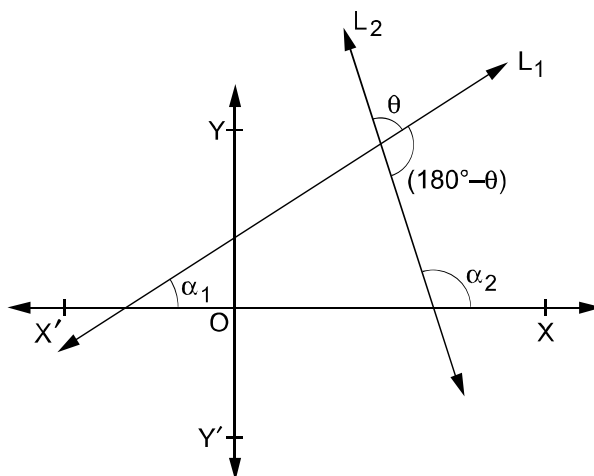
ANGLE BETWEEN TWO NON-VERTICAL AND NON-PERPENDICULAR LINES

THEOREM 4 If α is the acute angle between two non-vertical and non-perpendicular lines L_1 and L_2 with slopes m_1 and m_2 then prove that

$$\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|, \text{ where } 1 + m_1 m_2 \neq 0.$$

PROOF Let L_1 and L_2 be the two given non-vertical and non-perpendicular lines with slopes m_1 and m_2 respectively. Let α_1 and α_2 be their respective inclinations. Then,

$$m_1 = \tan \alpha_1 \text{ and } m_2 = \tan \alpha_2.$$



Let θ and $(180^\circ - \theta)$ be the angles between L_1 and L_2 .

From the given figure, we have

$$\theta = \alpha_2 - \alpha_1, \text{ where } \alpha_1, \alpha_2 \neq 90^\circ$$

$$\Rightarrow \tan \theta = \tan (\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}$$

$$\Rightarrow \tan \theta = \frac{(m_2 - m_1)}{(1 + m_1 m_2)} \quad \dots (i), \quad \text{where } 1 + m_1 m_2 \neq 0.$$

$$\text{Also, } \tan (180^\circ - \theta) = -\tan \theta = -\frac{(m_2 - m_1)}{(1 + m_1 m_2)} \quad \dots (ii)$$

$$\text{Thus, tangent of the angle between given lines} = \pm \frac{(m_2 - m_1)}{(1 + m_1 m_2)}$$

[from (i) and (ii)].

Let $\alpha = \min. \{\theta, (180^\circ - \theta)\}$, then α is acute and so $\tan \alpha > 0$.

$$\therefore \tan \alpha = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|.$$



EXAMPLE 16 Find the angle between the lines whose slopes are $\frac{1}{2}$ and 3.

SOLUTION Let θ be the angle between the given lines.

Let $m_1 = \frac{1}{2}$ and $m_2 = 3$. Then,

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{\left(3 - \frac{1}{2} \right)}{\left(1 + 3 \cdot \frac{1}{2} \right)} \right| = \left| \frac{(5/2)}{(5/2)} \right| = |1| = 1$$

$$\Rightarrow \theta = 45^\circ.$$

Hence, the angle between the given lines is 45° .

EXAMPLE 17 If $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$ be the vertices of a $\triangle ABC$, show that

$$\tan B = \frac{2}{3}.$$

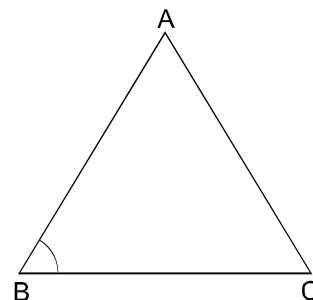
SOLUTION Clearly, B is the angle between BA and BC .

$$\text{Now, } m_1 = \text{slope of } BA = \frac{3 - 1}{2 + 2} = \frac{1}{2}$$

$$\text{and } m_2 = \text{slope of } BC = \frac{-4 - 3}{-2 - 2} = \frac{7}{4}.$$

$$\therefore \tan B = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \cdot \frac{1}{2}} \right| = \frac{2}{3}.$$

$$\text{Hence, } \tan B = \frac{2}{3}.$$



EXAMPLE 18 If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

SOLUTION We know that the acute angle θ between two lines having slopes m_1 and m_2 is given by

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|. \quad \dots (i)$$

$$\text{Let } m_1 = \frac{1}{2}, m_2 = m \text{ and } \theta = \frac{\pi}{4}.$$

Putting these values in (i), we get

$$\left| \frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} \right| = \tan \frac{\pi}{4} \Leftrightarrow \left| \frac{2m - 1}{2 + m} \right| = 1$$



$$\begin{aligned}
&\Leftrightarrow \left(\frac{2m-1}{2+m} \right) = \pm 1 \\
&\Leftrightarrow \frac{2m-1}{2+m} = 1 \quad \text{or} \quad \frac{2m-1}{2+m} = -1 \\
&\Leftrightarrow (2m-1) = (2+m) \quad \text{or} \quad (2m-1) = (-2-m) \\
&\Leftrightarrow m = 3 \quad \text{or} \quad m = \frac{-1}{3} .
\end{aligned}$$

Hence, the slope of the other line is 3 or $\frac{-1}{3}$.

EXERCISE 20B

- Find the slope of a line whose inclination is
 - 30°
 - 120°
 - 135°
 - 90°
- Find the inclination of a line whose slope is
 - $\sqrt{3}$
 - $\frac{1}{\sqrt{3}}$
 - 1
 - 1
 - $-\sqrt{3}$
- Find the slope of a line which passes through the points
 - (0, 0) and (4, -2)
 - (0, -3) and (2, 1)
 - (2, 5) and (-4, -4)
 - (-2, 3) and (4, -6)
- If the slope of the line joining the points $A(x, 2)$ and $B(6, -8)$ is $\frac{-5}{4}$, find the value of x .
- Show that the line through the points (5, 6) and (2, 3) is parallel to the line through the points (9, -2) and (6, -5).
- Find the value of x so that the line through (3, x) and (2, 7) is parallel to the line through (-1, 4) and (0, 6).
- Show that the line through the points (-2, 6) and (4, 8) is perpendicular to the line through the points (3, -3) and (5, -9).
- If $A(2, -5)$, $B(-2, 5)$, $C(x, 3)$ and $D(1, 1)$ be four points such that AB and CD are perpendicular to each other, find the value of x .
- Without using Pythagoras's theorem, show that the points $A(1, 2)$, $B(4, 5)$ and $C(6, 3)$ are the vertices of a right-angled triangle.
- Using slopes, show that the points $A(6, -1)$, $B(5, 0)$ and $C(2, 3)$ are collinear.
- Using slopes, find the value of x for which the points $A(5, 1)$, $B(1, -1)$ and $C(x, 4)$ are collinear.
- Using slopes, show that the points $A(-4, -1)$, $B(-2, -4)$, $C(4, 0)$ and $D(2, 3)$ taken in order, are the vertices of a rectangle.
- Using slopes, prove that the points $A(-2, -1)$, $B(1, 0)$, $C(4, 3)$ and $D(1, 2)$ are the vertices of a parallelogram.



14. If the three points $A(h, k)$, $B(x_1, y_1)$ and $C(x_2, y_2)$ lie on a line then show that

$$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$
15. If the points $A(a, 0)$, $B(0, b)$ and $P(x, y)$ are collinear, using slopes, prove that

$$\frac{x}{a} + \frac{y}{b} = 1.$$
16. A line passes through the points $A(4, -6)$ and $B(-2, -5)$. Show that the line AB makes an obtuse angle with the x -axis.
17. The vertices of a quadrilateral are $A(-4, 2)$, $B(2, 6)$, $C(8, 5)$ and $D(9, -7)$. Using slopes, show that the midpoints of the sides of the quad. $ABCD$ form a parallelogram.
18. Find the slope of the line which makes an angle of 30° with the positive direction of the y -axis, measured anticlockwise.
19. Find the angle between the lines whose slopes are $\sqrt{3}$ and $\frac{1}{\sqrt{3}}$.
20. Find the angle between the lines whose slopes are $(2 - \sqrt{3})$ and $(2 + \sqrt{3})$.
21. If $A(1, 2)$, $B(-3, 2)$ and $C(3, -2)$ be the vertices of a $\triangle ABC$, show that
 (i) $\tan A = 2$ (ii) $\tan B = \frac{2}{3}$ (iii) $\tan C = \frac{4}{7}$.
22. If θ is the angle between the lines joining the points $A(0, 0)$ and $B(2, 3)$, and the points $C(2, -2)$ and $D(3, 5)$, show that $\tan \theta = \frac{11}{23}$.
23. If θ is the angle between the diagonals of a parallelogram $ABCD$ whose vertices are $A(0, 2)$, $B(2, -1)$, $C(4, 0)$ and $D(2, 3)$. Show that $\tan \theta = 2$.
24. Show that the points $A(0, 6)$, $B(2, 1)$ and $C(7, 3)$ are three corners of a square $ABCD$. Find (i) the slope of the diagonal BD and (ii) the coordinates of the fourth vertex D .
25. $A(1, 1)$, $B(7, 3)$ and $C(3, 6)$ are the vertices of a $\triangle ABC$. If D is the midpoint of BC and $AL \perp BC$, find the slopes of (i) AD and (ii) AL .

ANSWERS (EXERCISE 20B)

1. (i) $\frac{1}{\sqrt{3}}$ (ii) $-\sqrt{3}$ (iii) -1 (iv) not defined
2. (i) 60° (ii) 30° (iii) 45° (iv) 135° (v) 120°
3. (i) $-\frac{1}{2}$ (ii) 2 (iii) $\frac{3}{2}$ (iv) $-\frac{3}{2}$ 4. $x = -2$ 6. $x = 9$
8. $x = 6$ 11. $x = 11$ 18. $-\sqrt{3}$ 19. 30°
20. 60° 24. (i) $\frac{7}{3}$ (ii) $(5, 8)$ 25. (i) $\frac{7}{8}$ (ii) $\frac{4}{3}$



HINTS TO SOME SELECTED QUESTIONS

11. For collinearity, we must have, slope of AB = slope of BC .
12. Let m_1, m_2, m_3, m_4 be the slopes of AB, BC, CD and DA respectively. Then, show that $m_1 = m_3, m_2 = m_4$ and $m_1 m_2 = -1$.
14. Slope of AB = slope of BC
$$\Rightarrow \frac{(y_1 - k)}{(x_1 - h)} = \frac{(y_2 - y_1)}{(x_2 - x_1)} \Rightarrow \frac{(k - y_1)}{(h - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\Rightarrow (h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$
15. Since the points A, B, P are collinear, we have
(slope of AB = slope of BP) $\Rightarrow \frac{b - 0}{0 - a} = \frac{y - b}{x - 0}$
$$\Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$$
16. Slope of AB is negative.
17. Midpoints of AB, BC, CD and DA are $P(-1, 4), Q\left(5, \frac{11}{2}\right), R\left(\frac{17}{2}, -1\right)$ and $S\left(\frac{5}{2}, \frac{-5}{2}\right)$.
Now, show that (slope of PQ = slope of $RS = \frac{1}{4}$) and (slope of QR = slope of $PS = \frac{-13}{7}$).
18. The given line makes an angle of $(90^\circ + 30^\circ) = 120^\circ$ with the positive direction of the x -axis. Hence, $m = \tan 120^\circ = -\sqrt{3}$.
24. $BD \perp AC$.
-

VARIOUS FORMS OF EQUATIONS OF A LINE

EQUATION OF A LINE

The equation of a straight line is the linear relation between two variables x and y , which is satisfied by the coordinates of each and every point on the line and not by those of any other point in the cartesian plane.

SOME ELEMENTARY RESULTS

(i) EQUATION OF X-AXIS:

We know that the ordinate of each point on the x -axis is 0. If $P(x, y)$ is any point on the x -axis, then $y = 0$.

Hence, the equation of x -axis is $y = 0$.

(ii) EQUATION OF Y-AXIS:

We know that the abscissa of each point on the y -axis is 0. If $P(x, y)$ is any point on the y -axis, then $x = 0$.

Hence, the equation of y -axis is $x = 0$.

(iii) EQUATION OF A LINE PARALLEL TO Y-AXIS:

Let AB be a straight line parallel to the y -axis lying on its right-hand side at a distance a from it.

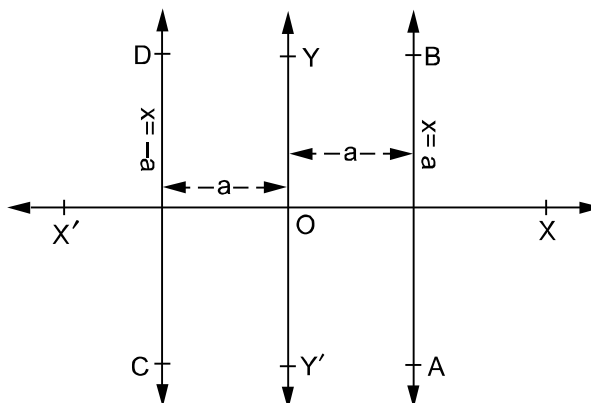


Then, the abscissa of each point on AB is a .

If $P(x, y)$ is any point on AB then $x = a$.

Thus, the equation of a vertical line at a distance a from the y -axis, lying on its right-hand side is $x = a$.

Similarly, the equation of a vertical line at a distance a from the y -axis, lying on its left-hand side is $x = -a$.



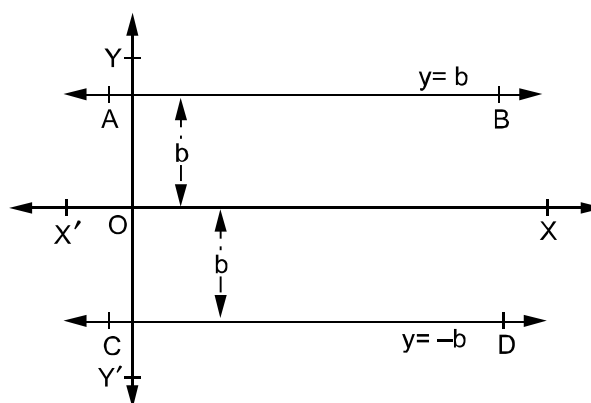
(iv) EQUATION OF A LINE PARALLEL TO X-AXIS:

Let AB be a straight line parallel to the x -axis, lying above it at a distance b from it.

Then, the ordinate of each point on AB is b .

If $P(x, y)$ is any point on AB , then $y = b$.

Thus, the equation of a horizontal line at a distance b from the x -axis, lying above the x -axis is $y = b$.



Similarly, the equation of a horizontal line at a distance b from the x -axis, lying below the x -axis is $y = -b$.

SUMMARY

- (i) Equation of x -axis is $y = 0$.
- (ii) Equation of y -axis is $x = 0$.
- (iii) Equation of a vertical line on RHS of the y -axis at a distance a from it is $x = a$.
- (iv) Equation of a vertical line on LHS of the x -axis at a distance a from it is $x = -a$.
- (v) Equation of a horizontal line lying above the x -axis at a distance b from it is $y = b$.
- (vi) Equation of a horizontal line lying below the x -axis at a distance b from it is $y = -b$.

EXAMPLE 1 Write down the equation of (i) x -axis (ii) y -axis.

SOLUTION We know that

- (i) the equation of x -axis is $y = 0$.
- (ii) the equation of y -axis is $x = 0$.



EXAMPLE 2 Write down the equation of a line parallel to the x -axis

- (i) at a distance of 5 units above the x -axis,
- (ii) at a distance of 4 units below the x -axis.

SOLUTION (i) The equation of a line parallel to the x -axis at a distance of 5 units above x -axis is $y = 5$.

(ii) The equation of a line parallel to the x -axis at a distance of 4 units below the x -axis is $y = -4$.

EXAMPLE 3 Write down the equation of a line parallel to the y -axis

- (i) at a distance of 7 units on left-hand side of the y -axis,
- (ii) at a distance of 3 units on right-hand side of the y -axis.

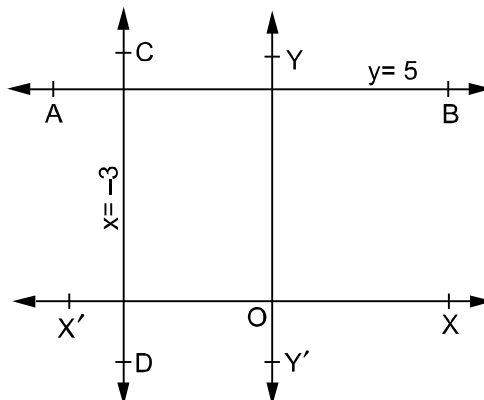
SOLUTION (i) The equation of a line parallel to the y -axis at a distance of 7 units on its left is $x = -7$.

(ii) The equation of a line parallel to the y -axis at a distance of 3 units on its right is $x = 3$.

EXAMPLE 4 Find the equations of the lines parallel to the axes and passing through the point $(-3, 5)$.

SOLUTION We know that

- (i) the equation of a line parallel to the x -axis and passing through $(-3, 5)$ is $y = 5$.
- (ii) the equation of a line parallel to the y -axis and passing through $(-3, 5)$ is $x = -3$.



EXAMPLE 5 Find the values of k for which the line

$$(k - 3)x + (k^2 - 4)y + (k - 1)(k - 6) = 0$$

- (i) is parallel to the x -axis,
- (ii) is parallel to the y -axis,
- (iii) passes through the origin.

SOLUTION Given equation of the line is $(k - 3)x + (k^2 - 4)y + (k - 1)(k - 6) = 0$.

- (i) The given line will be parallel to the x -axis, when the coefficient of y is zero,
i.e., when $k^2 - 4 = 0$.
This happens when $k = \pm 2$.
- (ii) The given line will be parallel to the y -axis, when the coefficient of x is zero,
i.e., when $k - 3 = 0$ or $k = 3$.



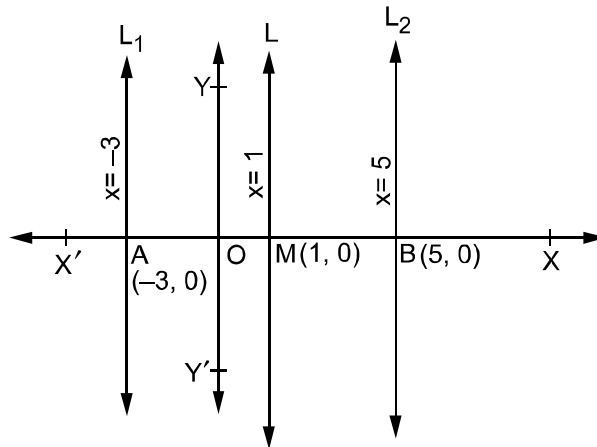
(iii) The given line will pass through the origin, if $(0, 0)$ satisfies the given equation,

i.e., when $(k - 1)(k - 6) = 0$.

This happens when $k = 1$ or $k = 6$.

EXAMPLE 6 Find the equation of a line which is equidistant from the lines $x = -3$ and $x = 5$.

SOLUTION Let $X'OX$ and YOY' be the coordinate axes



Let L_1 be the line $x = -3$. Then, it is parallel to the y -axis.

It meets the x -axis at $A(-3, 0)$.

Let L_2 be the line $x = 5$. Then, it is parallel to the y -axis.

It meets the x -axis at $B(5, 0)$.

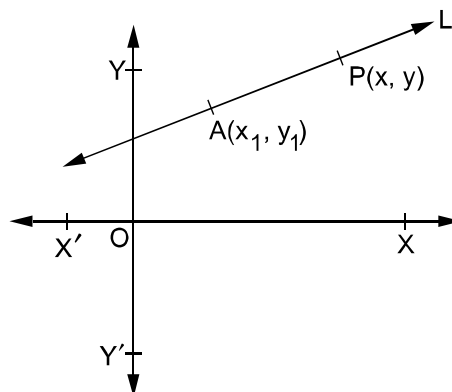
Midpoint of AB is $M\left(\frac{-3 + 5}{2}, \frac{0 + 0}{2}\right)$, i.e., $M(1, 0)$.

So, the line $x = 1$ is equidistant from L_1 and L_2 .

EQUATION OF A LINE IN POINT-SLOPE FORM

THEOREM 1 Prove that the equation of a non-vertical line with slope m and passing through the point (x_1, y_1) is $(y - y_1) = m(x - x_1)$.

SOLUTION Let L be a non-vertical line with slope m and passing through a point $A(x_1, y_1)$.



Let $P(x, y)$ be an arbitrary point on L .

Then, slope of line $L = \frac{(y - y_1)}{(x - x_1)}$.

$$\therefore \frac{(y - y_1)}{(x - x_1)} = m \Leftrightarrow (y - y_1) = m(x - x_1).$$

Hence, the required equation is, $(y - y_1) = m(x - x_1)$.

EXAMPLE 1 Find the equation of a line passing through the point $(4, 3)$ and having slope 2.

SOLUTION We know that the equation of a line with slope m and passing through the point (x_1, y_1) is given by

$$(y - y_1) = m(x - x_1).$$

Here, $m = 2$, $x_1 = 4$ and $y_1 = 3$.

Hence, the required equation is

$$(y - 3) = 2(x - 4), \text{ i.e., } 2x - y - 5 = 0.$$

EXAMPLE 2 Find the equation of a line which makes an angle of 135° with the x -axis and passes through the point $(3, 5)$.

SOLUTION Here, $m = \tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1$.

Hence, the required equation is

$$\frac{y - 5}{x - 3} = -1 \Leftrightarrow (y - 5) = 3 - x \Leftrightarrow x + y - 8 = 0.$$

EXAMPLE 3 Find the equation of a line passing through the point $(3, -4)$ and parallel to the x -axis.

SOLUTION Since the given line is parallel to the x -axis, we have $m = 0$.

So, the required equation is

$$\frac{y + 4}{y - 3} = 0 \Leftrightarrow y + 4 = 0.$$

EQUATION OF A LINE IN TWO-POINT FORM

THEOREM 2 Prove that the equation of a non-vertical line passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$\frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}.$$

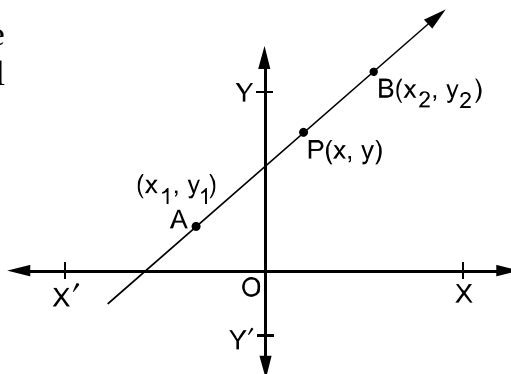
PROOF Let L be the line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ and let $P(x, y)$ be an arbitrary point on L .

Then, clearly we have

slope of AP = slope of AB

$$\Rightarrow \frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)},$$

which is the required equation.



EXAMPLE 4 Find the equation of a line passing through the points $(-1, 1)$ and $(2, -4)$.

SOLUTION Let the given points be $A(-1, 1)$ and $B(2, -4)$.

We know that the equation of a line passing through the points

$$(x_1, y_1) \text{ and } (x_2, y_2) \text{ is given by } \frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}.$$

Here, $x_1 = -1$, $y_1 = 1$, $x_2 = 2$ and $y_2 = -4$.

So, the equation of a line passing through the given points is

$$\begin{aligned} \frac{(y - 1)}{(x - (-1))} &= \frac{(-4 - 1)}{2 - (-1)} \Leftrightarrow \frac{(y - 1)}{(x + 1)} = \frac{-5}{3} \\ &\Leftrightarrow 3(y - 1) = -5(x + 1) \\ &\Leftrightarrow 5x + 3y + 2 = 0. \end{aligned}$$

Hence, the required equation is $5x + 3y + 2 = 0$.

EXAMPLE 5 Show that the three points $(3, 0)$, $(-2, -2)$ and $(8, 2)$ are collinear. Also, find the equation of the straight line on which these points lie.

SOLUTION Let $A(3, 0)$, $B(-2, -2)$ and $C(8, 2)$ be the given points.

Then, the equation of line AB is given by

$$\begin{aligned} \frac{y - 0}{x - 3} &= \frac{-2 - 0}{-2 - 3} \quad \left[\text{using } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \right] \\ \Rightarrow \frac{y}{x - 3} &= \frac{2}{5} \\ \Rightarrow 5y &= 2x - 6 \end{aligned} \quad \dots (i)$$

$\Rightarrow 2x - 5y - 6 = 0$, which is the required equation.

Putting $x = 8$ and $y = 2$ in (i), we get

$$\text{LHS} = (2 \times 8) - (5 \times 2) - 6 = 0 = \text{RHS}.$$

Thus, the point $C(8, 2)$ also lies on (i).

Hence, the given points lie on the same straight line, whose equation is $2x - 5y - 6 = 0$.

EXAMPLE 6 Show that the points $(a, 0)$, $(0, b)$ and $(3a, -2b)$ are collinear. Also find the equation of the line containing them.

SOLUTION Let $A(a, 0)$, $B(0, b)$ and $C(3a, -2b)$ be the given points.

Then, the equation of line AB is given by

$$\begin{aligned} \frac{y - 0}{x - a} &= \frac{b - 0}{0 - a} \Rightarrow -ay = bx - ab \\ &\Rightarrow bx + ay - ab = 0. \end{aligned}$$

Thus, the equation of line AB is $bx + ay - ab = 0$ (i)

Putting $x = 3a$ and $y = -2b$ in (i), we get

$$\text{LHS} = b \cdot 3a + a(-2b) - ab = 3ab - 2ab - ab = 0 = \text{RHS}.$$

Thus, the point $C(3a, -2b)$ also lies on AB .

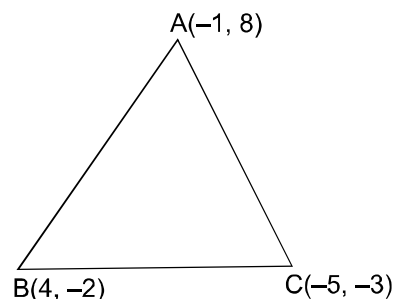


Hence, the given points are collinear and the equation of the line containing them is $bx + ay - ab = 0$.

EXAMPLE 7 Find the equations of the sides of a triangle whose vertices are $A(-1, 8)$, $B(4, -2)$ and $C(-5, -3)$.

SOLUTION Here, we make use of the equation of a line in two-point form.

$$\begin{aligned}\text{Equation of } AB \text{ is } \frac{y-8}{x+1} &= \frac{-2-8}{4+1} \\ \Rightarrow 5(y-8) + 10(x+1) &= 0 \\ \Rightarrow 10x + 5y - 30 &= 0 \\ \Rightarrow 2x + y - 6 &= 0;\end{aligned}$$



$$\text{equation of } BC \text{ is } \frac{y+2}{x-4} = \frac{-3+2}{-5-4} \Rightarrow -9(y+2) = -(x-4)$$

$$\Rightarrow x - 9y - 22 = 0;$$

$$\text{equation of } AC \text{ is } \frac{y-8}{x+1} = \frac{-3-8}{-5+1} \Rightarrow -4(y-8) = -11(x+1)$$

$$\Rightarrow 11x - 4y + 43 = 0.$$

EXAMPLE 8 Find the equations of the medians of a $\triangle ABC$ whose vertices are $A(2, 5)$, $B(-4, 9)$ and $C(-2, -1)$.

SOLUTION The vertices of $\triangle ABC$ are $A(2, 5)$, $B(-4, 9)$ and $C(-2, -1)$.

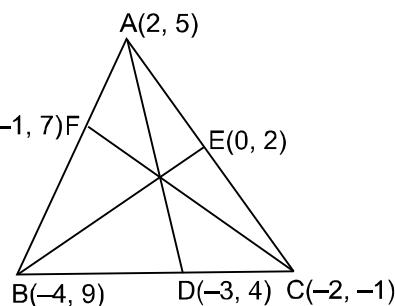
Let D, E, F be the midpoints of BC, CA and AB respectively.

Then, these points are

$$D\left(\frac{-4-2}{2}, \frac{9-1}{2}\right), E\left(\frac{-2+2}{2}, \frac{-1+5}{2}\right) \quad (-1, 7)F$$

$$\text{and } F\left(\frac{2-4}{2}, \frac{5+9}{2}\right)$$

i.e., $D(-3, 4)$, $E(0, 2)$ and $F(-1, 7)$.



Thus, AD, BE and CF are the medians of $\triangle ABC$.

$$\begin{aligned}\text{Equation of median } AD \text{ is } \frac{y-5}{x-2} &= \frac{4-5}{-3-2} \Rightarrow \frac{y-5}{x-2} = \frac{1}{5} \\ \Rightarrow 5(y-5) &= x-2 \\ \Rightarrow x - 5y + 23 &= 0.\end{aligned}$$

$$\begin{aligned}\text{Equation of median } BE \text{ is } \frac{y-9}{x+4} &= \frac{2-9}{0+4} \Rightarrow \frac{y-9}{x+4} = \frac{-7}{4} \\ \Rightarrow 4(y-9) + 7(x+4) &= 0 \\ \Rightarrow 7x + 4y - 8 &= 0.\end{aligned}$$

Equation of median CF is $\frac{y+1}{x+2} = \frac{7+1}{-1+2} \Rightarrow (y+1) = 8(x+2)$
 $\Rightarrow 8x - y + 15 = 0.$

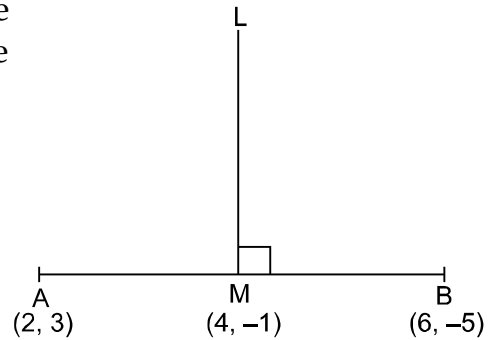
EXAMPLE 9 Find the equation of the perpendicular bisector of the line joining the points $A(2, 3)$ and $B(6, -5)$.

SOLUTION Here, $A(2, 3)$ and $B(6, -5)$ are the end points of the given line segment.

The midpoint of AB is

$$M\left(\frac{2+6}{2}, \frac{3-5}{2}\right), \text{ i.e., } M(4, -1).$$

$$\text{Slope of } AB = \frac{-5-3}{6-2} = -2$$



Let $LM \perp AB$ and let the slope of LM be m .

$$\text{Then, } m \times (-2) = -1 \Rightarrow m = \frac{1}{2} \quad [\because LM \perp AB].$$

Clearly, LM is the perpendicular bisector of AB .

Now, LM is a line with slope $= \frac{1}{2}$ and passing through $M(4, -1)$.

So, the required equation is

$$\frac{y - (-1)}{x - 4} = \frac{1}{2} \Rightarrow 2(y + 1) = (x - 4)$$

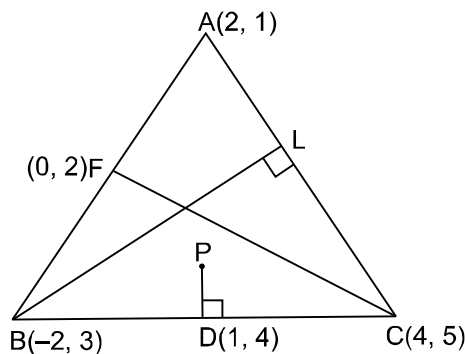
$$\Rightarrow x - 2y - 6 = 0.$$

EXAMPLE 10 If $A(2, 1)$, $B(-2, 3)$ and $C(4, 5)$ are the vertices of a $\triangle ABC$ then find the equation of

- (i) the median through C (ii) the altitude through B
 (iii) the right bisector of side BC .

SOLUTION (i) Let F be the midpoint of side AB .

Then, CF is the median through C .



$$\text{Coordinates of } F \text{ are } F\left(\frac{2+(-2)}{2}, \frac{1+3}{2}\right), \text{ i.e., } F(0, 2).$$

So, the equation of median CF is given by

$$\frac{y-5}{x-4} = \frac{2-5}{0-4} \Leftrightarrow \frac{y-5}{x-4} = \frac{3}{4}$$

$$\Leftrightarrow 4(y-5) = 3(x-4) \Leftrightarrow 3x - 4y + 8 = 0.$$

Hence, the equation of median CF is $3x - 4y + 8 = 0$.

(ii) Draw $BL \perp AC$. Then, BL is the altitude through B .

$$\text{Slope of } AC = \frac{5-1}{4-2} = 2$$

Let the slope BL be m .

Since $BL \perp AC$, we have $2m = -1$ and therefore, $m = \frac{-1}{2}$.

Thus, the slope of BL is $\frac{-1}{2}$.

So, the equation of BL is given by

$$\frac{y-3}{x+2} = \frac{-1}{2} \Leftrightarrow 2(y-3) = -(x+2) \Leftrightarrow x + 2y - 4 = 0.$$

Hence, the equation of altitude BL is $x + 2y - 4 = 0$.

(iii) Let D be the midpoint of BC .

Then, the coordinates of D are $D\left(\frac{-2+4}{2}, \frac{3+5}{2}\right)$, i.e., $D(1, 4)$.

Through D , draw $DP \perp BC$.

$$\text{Slope of } BC = \frac{5-3}{4+2} = \frac{2}{6} = \frac{1}{3}.$$

Let the slope of PD be m .

Since $PD \perp BC$, we have $m \times \frac{1}{3} = -1 \Rightarrow m = -3$.

So, the slope of PD is -3 .

So, the equation of PD is given by

$$\frac{y-4}{x-1} = -3 \Leftrightarrow -3(x-1) = (y-4) \Leftrightarrow 3x + y - 7 = 0.$$

Hence, the equation of the right bisector of BC is $3x + y - 7 = 0$.

EXAMPLE 11 Find the equation of the bisector of $\angle A$ of $\triangle ABC$, whose vertices are $A(-2, 4)$, $B(5, 5)$ and $C(4, -2)$.

SOLUTION Let AD be the bisector of $\angle A$.

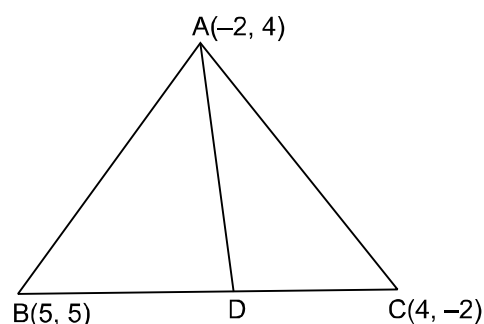
Then, $BD : DC = AB : AC$.

$$\text{Now, } |AB| = \sqrt{(5-4)^2 + (5+2)^2}$$

$$= \sqrt{50} = 5\sqrt{2}.$$

$$\text{and } |AC| = \sqrt{(-2-4)^2 + (4+2)^2}$$

$$= \sqrt{72} = 6\sqrt{2}.$$



$$\therefore \frac{BD}{DC} = \frac{AB}{AC} = \frac{5\sqrt{2}}{6\sqrt{2}} = \frac{5}{6} \Rightarrow BD : DC = 5 : 6.$$

Thus, D divides BC in the ratio $5 : 6$.

So, the coordinates of D are $\left(\frac{5 \times 4 + 6 \times 5}{5 + 6}, \frac{5 \times (-2) + 6 \times 5}{5 + 6} \right)$,

$$\text{i.e., } D\left(\frac{50}{11}, \frac{20}{11}\right)$$

$\therefore$ equation of line AD is given by

$$\frac{y - 4}{x + 2} = \frac{\frac{20}{11} - 4}{\frac{50}{11} + 2} \Leftrightarrow \frac{y - 4}{x + 2} = \frac{-24}{72} = \frac{-1}{3}$$

$$\Leftrightarrow 3(y - 4) = -(x + 2) \Leftrightarrow x + 3y - 10 = 0.$$

Hence, the equation of the bisector of $\angle A$ is $x + 3y - 10 = 0$.

EXERCISE 20C

- Find the equation of a line parallel to the x -axis at a distance of
(i) 4 units above it (ii) 5 units below it.
- Find the equation of a line parallel to the y -axis at a distance of
(i) 6 units to its right (ii) 3 units to its left.
- Find the equation of a line parallel to the x -axis and having intercept -3 on the y -axis.
- Find the equation of a horizontal line passing through the point $(4, -2)$.
- Find the equation of a vertical line passing through the point $(-5, 6)$.
- Find the equation of a line which is equidistant from the lines $x = -2$ and $x = 6$.
- Find the equation of a line which is equidistant from the lines $y = 8$ and $y = -2$.
- Find the equation of a line
(i) whose slope is 4 and which passes through the point $(5, -7)$;
(ii) whose slope is -3 and which passes through the point $(-2, 3)$;
(iii) which makes an angle of $\frac{2\pi}{3}$ with the positive direction of the x -axis and passes through the point $(0, 2)$.
- Find the equation of a line whose inclination with the x -axis is 30° and which passes through the point $(0, 5)$.
- Find the equation of a line whose inclination with the x -axis is 150° and which passes through the point $(3, -5)$.
- Find the equation of a line passing through the origin and making an angle of 120° with the positive direction of the x -axis.



12. Find the equation of a line which cuts off intercept 5 on the x -axis and makes an angle of 60° with the positive direction of the x -axis.
13. Find the equation of the line passing through the point $P(4, -5)$ and parallel to the line joining the points $A(3, 7)$ and $B(-2, 4)$.
14. Find the equation of the line passing through the point $P(-3, 5)$ and perpendicular to the line passing through the points $A(2, 5)$ and $B(-3, 6)$.
15. Find the slope and the equation of the line passing through the points:
 - (i) $(3, -2)$ and $(-5, -7)$
 - (ii) $(-1, 1)$ and $(2, -4)$
 - (iii) $(5, 3)$ and $(-5, -3)$
 - (iv) (a, b) and $(-a, b)$
16. Find the angle which the line joining the points $(1, \sqrt{3})$ and $(\sqrt{2}, \sqrt{6})$ makes with the x -axis.
17. Prove that the points $A(1, 4)$, $B(3, -2)$ and $C(4, -5)$ are collinear. Also find the equation of the line on which these points lie.
18. If $A(0, 0)$, $B(2, 4)$ and $C(6, 4)$ are the vertices of a $\triangle ABC$, find the equations of its sides.
19. If $A(-1, 6)$, $B(-3, -9)$ and $C(5, -8)$ are the vertices of a $\triangle ABC$, find the equations of its medians.
20. Find the equation of the perpendicular bisector of the line segment whose end points are $A(10, 4)$ and $B(-4, 9)$.
21. Find the equations of the altitudes of a $\triangle ABC$, whose vertices are $A(2, -2)$, $B(1, 1)$ and $C(-1, 0)$.
22. If $A(4, 3)$, $B(0, 0)$ and $C(2, 3)$ are the vertices of a $\triangle ABC$, find the equation of the bisector of $\angle A$.
23. The midpoints of the sides BC , CA and AB of a $\triangle ABC$ are $D(2, 1)$, $E(-5, 7)$ and $F(-5, -5)$ respectively. Find the equations of the sides of $\triangle ABC$.
24. If $A(1, 4)$, $B(2, -3)$ and $C(-1, -2)$ are the vertices of a $\triangle ABC$, find the equation of
 - (i) the median through A
 - (ii) the altitude through A
 - (iii) the perpendicular bisector of BC .

ANSWERS (EXERCISE 20C)

1. (i) $y - 4 = 0$ (ii) $y + 5 = 0$
2. (i) $x - 6 = 0$ (ii) $x + 3 = 0$
3. $y + 3 = 0$
4. $y + 2 = 0$
5. $x + 5 = 0$
6. $x = 2$
7. $y = 3$
8. (i) $4x - y - 27 = 0$ (ii) $3x + y + 3 = 0$ (iii) $\sqrt{3}x + y - 2 = 0$
9. $x - \sqrt{3}y + 5\sqrt{3} = 0$
10. $x + \sqrt{3}y + (-3 + 5\sqrt{3}) = 0$
11. $\sqrt{3}x + y = 0$
12. $\sqrt{3}x - y - 5\sqrt{3} = 0$
13. $3x - 5y - 37 = 0$
14. $5x - y + 20 = 0$
15. (i) $\frac{5}{8}, 5x - 8y - 31 = 0$ (ii) $\frac{-5}{3}, 5x + 3y + 2 = 0$ (iii) $\frac{3}{5}, 3x - 5y = 0$ (iv) $0, y = b$
16. 60°
17. $3x + y - 7 = 0$
18. $y = 4, 2x - 3y = 0, 2x - y = 0$



19. $29x + 4y + 5 = 0$, $8x - 5y - 21 = 0$, $13x + 14y + 47 = 0$ 20. $28x - 10y - 19 = 0$
 21. $2x + y - 2 = 0$, $3x - 2y - 1 = 0$, $x - 3y + 1 = 0$ 22. $x - 3y + 5 = 0$
 23. $x - 2 = 0$, $6x - 7y + 79 = 0$, $6x + 7y + 65 = 0$
 24. (i) $13x - y - 9 = 0$ (ii) $3x - y + 1 = 0$ (iii) $3x - y - 4 = 0$

HINTS TO SOME SELECTED QUESTIONS

6. Midpoint of $A(-2, 0)$ and $B(6, 0)$ is $M\left(\frac{-2+6}{2}, 0\right)$, i.e., $M(2, 0)$.

So, the required line is $x = 2$.

7. Midpoint of $A(0, 8)$ and $B(0, -2)$ is $M\left(0, \frac{8-2}{2}\right)$, i.e., $M(0, 3)$.

So, the required line is $y = 3$.

12. The given line passes through the point $(5, 0)$ and $m = \tan 60^\circ = \sqrt{3}$. So, the required equation is $\frac{y-0}{x-5} = \sqrt{3} \Leftrightarrow y = \sqrt{3}(x-5)$.

13. Slope of the given line = slope of $AB = \frac{4-7}{-2-3} = \frac{3}{5}$. Thus, $m = \frac{3}{5}$.

$\therefore$ required equation is $\frac{y+5}{x-4} = \frac{3}{5} \Leftrightarrow 3x - 5y - 37 = 0$.

14. Slope of $AB = m_1 = \frac{6-5}{-3-2} = \frac{-1}{5}$.

Now, $m_1 m_2 = -1 \Leftrightarrow \frac{-1}{5} \times m_2 = -1 \Leftrightarrow m_2 = 5$.

$\therefore$ required equation is $\frac{y-5}{x+3} = 5 \Leftrightarrow 5x - y + 20 = 0$.

16. $m = \tan \theta = \left(\frac{\sqrt{6}-\sqrt{3}}{\sqrt{2}-1}\right) = \sqrt{3} \Leftrightarrow \theta = 60^\circ$.



EQUATION OF A LINE IN SLOPE-INTERCEPT FORM

THEOREM 1 Prove that the equation of a line with slope m and making an intercept c on the y -axis, is given by $y = mx + c$.

PROOF Let L be a given line with slope m , making an intercept c on the y -axis. Then, L cuts the y -axis at a point C such that $OC = c$.

Clearly, the coordinates of C are $(0, c)$.

Thus, the line L has slope m and passes through the point $C(0, c)$.

So, by the point-slope form, the equation of L is

$$(y - c) = m(x - 0) \Rightarrow y = mx + c.$$

Hence, $y = mx + c$ is the required equation.

REMARK The equation of a line with slope m and y -intercept c is given by $y = mx + c$.

THEOREM 2 Prove that the equation of a line with slope m and making an intercept d on the x -axis, is given by $y = m(x - d)$.

PROOF Let L be a given line with slope m , making an intercept d on the x -axis. Then, L cuts the x -axis at a point D such that $OD = d$.

Clearly, the coordinates of D are $(d, 0)$.

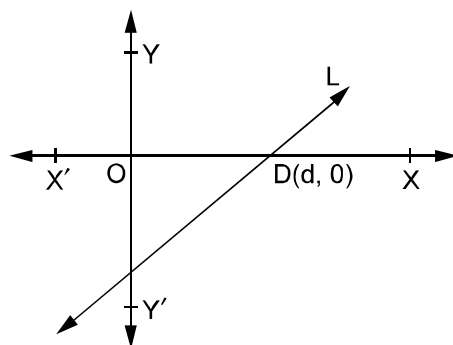
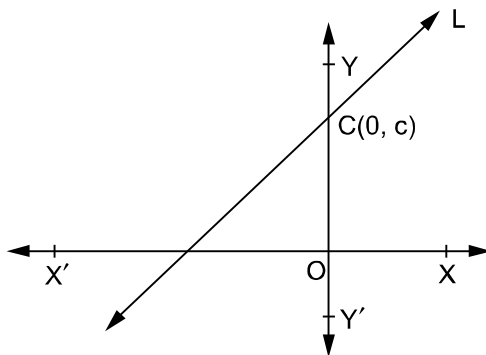
Thus, the line L has slope m and passes through the point $D(d, 0)$.

So, by the point-slope form, the equation of L is

$$\frac{(y - 0)}{(x - d)} = m \Leftrightarrow y = m(x - d).$$

Hence, $y = m(x - d)$ is the required equation.

REMARK The equation of a line with slope m and x -intercept d is given by $y = m(x - d)$.



SUMMARY

(i) The equation of a line with slope m and y -intercept c is given by

$$y = mx + c.$$

(ii) The equation of a line with slope m and x -intercept d is given by

$$y = m(x - d).$$

SOLVED EXAMPLES

EXAMPLE 1 Find the equation of a line whose slope is $\frac{1}{2}$ and y-intercept equal to $\frac{-5}{4}$.

SOLUTION We know that the equation of a line with slope m and y-intercept c is given by $y = mx + c$.

Here, $m = \frac{1}{2}$ and $c = \frac{-5}{4}$.

Hence, the required equation of the line is

$$y = \frac{1}{2}x - \frac{5}{4} \Rightarrow 4y = 2x - 5$$

$$\Rightarrow 2x - 4y - 5 = 0.$$

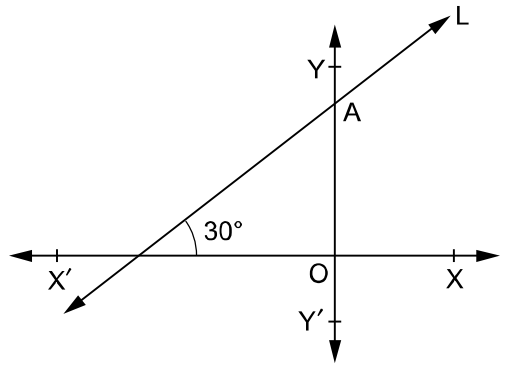
EXAMPLE 2 Find the equation of a line which intersects the y-axis at a distance of 3 units above the origin and makes an angle of 30° with the positive direction of the x-axis.

SOLUTION Here, $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$ and $c = 3$.

Hence, the required equation of the line is

$$y = \frac{1}{\sqrt{3}}x + 3 \Rightarrow \sqrt{3}y = x + 3\sqrt{3} \quad [\because y = mx + c]$$

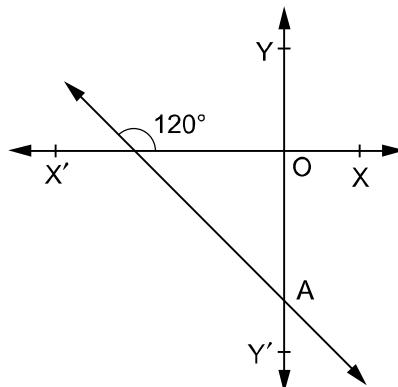
$$\Rightarrow x - \sqrt{3}y + 3\sqrt{3} = 0.$$



EXAMPLE 3 Find the equation of a line which cuts off an intercept of 4 units on negative direction of the y-axis and makes an angle of 120° with the positive direction of the x-axis.

SOLUTION We know that the equation of a line with slope m and y-intercept c is given by

$$y = mx + c.$$



Here, $m = \tan 120^\circ = \tan (180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}$ and $c = -4$.

Hence, the required equation of the line is

$$y = -\sqrt{3}x - 4 \Rightarrow \sqrt{3}x + y + 4 = 0 \quad [\because y = mx + c].$$

EXAMPLE 4 Find the equation of a line for which $\tan \theta = \frac{1}{3}$ and x -intercept equal to 5 units.

SOLUTION We know that the equation of a line with slope m and x -intercept d is given by $y = m(x - d)$.

Here, $m = \tan \theta = \frac{1}{3}$ and $d = 5$.

Hence, the required equation of the line is

$$\begin{aligned} y &= \frac{1}{3}(x - 5) \Rightarrow 3y = x - 5 \\ &\Rightarrow x - 3y - 5 = 0. \end{aligned}$$

EXAMPLE 5 Find the equation of a line which cuts the x -axis at a distance of 3 units to the left of the origin and has a slope equal to -2 .

SOLUTION We know that the equation of a line with slope m and x -intercept d is given by $y = m(x - d)$.

Here, $m = -2$ and $d = -3$.

Hence, the required equation of the line is

$$y = -2(x + 3) \Rightarrow 2x + y + 6 = 0.$$

EXAMPLE 6 Reduce the equation $6x + 3y - 5 = 0$ to the slope-intercept form and find its slope and y -intercept.

SOLUTION We know that the equation of a line with slope m and y -intercept c is given by $y = mx + c$.

$$\begin{aligned} \text{Now, } 6x + 3y - 5 &= 0 \Rightarrow 3y = -6x + 5 \\ &\Rightarrow y = -2x + \frac{5}{3}. \end{aligned}$$

$$\therefore m = -2 \text{ and } c = \frac{5}{3}.$$

Hence, slope $= -2$ and y -intercept $= \frac{5}{3}$.

EXAMPLE 7 Prove that the lines $x + 2y - 9 = 0$ and $2x + 4y + 5 = 0$ are parallel.

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

$$\begin{aligned} \text{Then, } x + 2y - 9 &= 0 \Rightarrow 2y = -x + 9 \\ &\Rightarrow y = -\frac{1}{2}x + \frac{9}{2}. \end{aligned}$$

$$\begin{aligned} \text{And, } 2x + 4y + 5 &= 0 \Rightarrow 4y = -2x - 5 \\ &\Rightarrow y = -\frac{1}{2}x - \frac{5}{4}. \end{aligned}$$



$$\therefore m_1 = -\frac{1}{2} \text{ and } m_2 = -\frac{1}{2}.$$

Thus, $m_1 = m_2$.

Hence, the given lines are parallel.

EXAMPLE 8 Show that the lines $27x - 18y + 25 = 0$ and $2x + 3y + 7 = 0$ are perpendicular to each other.

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

$$\text{Then, } 27x - 18y + 25 = 0 \Rightarrow 18y = 27x + 25$$

$$\Rightarrow y = \frac{3}{2}x + \frac{25}{18}$$

$$\text{And, } 2x + 3y + 7 = 0 \Rightarrow 3y = -2x - 7$$

$$\Rightarrow y = -\frac{2}{3}x - \frac{7}{3}.$$

$$\therefore m_1 = \frac{3}{2} \text{ and } m_2 = -\frac{2}{3}$$

$$\Rightarrow m_1 m_2 = \left(\frac{3}{2}\right) \times \left(-\frac{2}{3}\right) = -1$$

Hence, the given lines are perpendicular to each other.

EXAMPLE 9 Find the angle made by the line $x + \sqrt{3}y - 6 = 0$ with the positive direction of the x -axis.

SOLUTION Let the required angle be θ .

$$\text{Now, } x + \sqrt{3}y - 6 = 0 \Rightarrow y = -\frac{1}{\sqrt{3}}x + \frac{6}{\sqrt{3}}.$$

$$\therefore m = -\frac{1}{\sqrt{3}} \Leftrightarrow \tan \theta = -\frac{1}{\sqrt{3}} = -\tan 30^\circ = \tan (180^\circ - 30^\circ) = \tan 150^\circ.$$

$$\therefore \theta = 150^\circ.$$

Hence, the given line makes an angle of 150° with the positive direction of the x -axis.

EXAMPLE 10 Find the angle made by the line $x \cos 30^\circ + y \sin 30^\circ + \sin 120^\circ = 0$ with the positive direction of the x -axis.

SOLUTION Let the required angle be θ .

$$\text{Now, } x \cos 30^\circ + y \sin 30^\circ + \sin 120^\circ = 0$$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}\right)x + \left(\frac{1}{2}\right)y + \frac{\sqrt{3}}{2} = 0$$

$$\Rightarrow \sqrt{3}x + y + \sqrt{3} = 0$$

$$\Rightarrow y = (-\sqrt{3})x + (-\sqrt{3})$$

$$\Rightarrow y = mx + c, \text{ where } m = -\sqrt{3} \text{ and } c = -\sqrt{3}.$$

$$\text{Now, } m = -\sqrt{3}$$

$$\Leftrightarrow \tan \theta = -\sqrt{3} = -\tan 60^\circ = \tan (180^\circ - 60^\circ) = \tan 120^\circ.$$



$$\therefore \theta = 120^\circ.$$

Hence, the given line makes an angle of 120° with the positive direction of the x -axis.

EXAMPLE 11 Find the angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

$$\text{Now, } \sqrt{3}x + y = 1 \Rightarrow y = -\sqrt{3}x + 1$$

$$\text{and } x + \sqrt{3}y = 1 \Rightarrow y = -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}.$$

$$\therefore m_1 = -\sqrt{3} \text{ and } m_2 = \frac{-1}{\sqrt{3}}.$$

Let θ be the angle between the given lines. Then,

$$\begin{aligned} \tan \theta &= \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \\ &= \left| \frac{\frac{-1}{\sqrt{3}} + \sqrt{3}}{\left\{ 1 + (-\sqrt{3}) \times \left(\frac{-1}{\sqrt{3}} \right) \right\}} \right| = \left| \frac{2}{\sqrt{3}} \times \frac{1}{2} \right| = \left| \frac{1}{\sqrt{3}} \right| = \frac{1}{\sqrt{3}} \end{aligned}$$

$$\Rightarrow \theta = 30^\circ \text{ and } (180^\circ - \theta) = (180^\circ - 30^\circ) = 150^\circ.$$

Hence, the angles between the given lines are 30° and 150° .

EXAMPLE 12 Show that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $b_1, b_2 \neq 0$ are (i) parallel, if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ (ii) perpendicular, if

$$a_1a_2 + b_1b_2 = 0.$$

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

$$\text{Now, } a_1x + b_1y + c_1 = 0 \Rightarrow y = \frac{-a_1}{b_1} \cdot x - \frac{c_1}{b_1}$$

$$\text{and, } a_2x + b_2y + c_2 = 0 \Rightarrow y = \frac{-a_2}{b_2} \cdot x - \frac{c_2}{b_2}.$$

$$\therefore m_1 = \frac{-a_1}{b_1} \text{ and } m_2 = \frac{-a_2}{b_2}.$$

(i) Given lines are parallel, if $m_1 = m_2$ which gives

$$\frac{-a_1}{b_1} = \frac{-a_2}{b_2} \quad \text{or} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

(ii) Given lines are perpendicular, if $m_1 m_2 = -1$ which gives

$$\left(\frac{-a_1}{b_1} \right) \cdot \left(\frac{-a_2}{b_2} \right) = -1 \quad \text{or} \quad a_1a_2 + b_1b_2 = 0.$$

EXAMPLE 13 Find the equation of the line passing through the point $(2, -5)$ and parallel to the line $2x - 3y = 7$.



SOLUTION We have

$$2x - 3y = 7 \Rightarrow y = \frac{2}{3}x - \frac{7}{3}.$$

$$\therefore \text{ slope of the given line} = \frac{2}{3}$$

$$\text{and slope of a line parallel to the given line} = \frac{2}{3}.$$

Now, the equation of a line with slope $\frac{2}{3}$ and passing through the point $(2, -5)$ is given by

$$\begin{aligned} \frac{y - (-5)}{x - 2} &= \frac{2}{3} \Rightarrow \frac{y + 5}{x - 2} = \frac{2}{3} & \left[\because \frac{(y - y_1)}{(x - x_1)} = m \right] \\ \Rightarrow 3(y + 5) &= 2(x - 2) \\ \Rightarrow 2x - 3y - 19 &= 0, \text{ which is the required equation.} \end{aligned}$$

EXAMPLE 14 Find the equation of the line passing through the point $(-2, -4)$ and perpendicular to the line $3x - y + 5 = 0$.

SOLUTION We have

$$3x - y + 5 = 0 \Rightarrow y = 3x + 5$$

$$\therefore \text{ slope of the given line} = 3 \quad \left[\because y = mx + c \right]$$

$$\text{and slope of a line perpendicular to the given line} = -\frac{1}{3}$$

$$\left[\because m_2 = -\frac{1}{m_1} \right].$$

Now, the equation of a line with slope $-\frac{1}{3}$ and passing through the point $(-2, -4)$ is given by

$$\begin{aligned} \frac{y - (-4)}{x - (-2)} &= -\frac{1}{3} \Rightarrow \frac{y + 4}{x + 2} = \frac{-1}{3} & \left[\because \frac{(y - y_1)}{(x - x_1)} = m \right] \\ \Rightarrow 3(y + 4) &= -(x + 2) \\ \Rightarrow x + 3y + 14 &= 0, \text{ which is the required equation.} \end{aligned}$$

EXAMPLE 15 Find the equation of the line whose y -intercept is -3 and which is perpendicular to the line $3x - 2y + 5 = 0$.

SOLUTION $3x - 2y + 5 = 0 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}$

$$\therefore \text{ slope of the given line} = \frac{3}{2}.$$

$$\therefore \text{ slope of a line perpendicular to the given line} = \frac{-2}{3} \left[\because m_2 = \frac{-1}{m_1} \right].$$



Now, the equation of a line with slope $\frac{-2}{3}$ and y -intercept -3 is given by

$$y = \frac{-2}{3}x - 3 \quad [\because y = mx + c]$$

$\Rightarrow 2x + 3y + 9 = 0$, which is the required equation.

EXAMPLE 16 Find the equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x -intercept 3.

SOLUTION $x - 7y + 5 = 0 \Rightarrow y = \frac{1}{7}x + \frac{5}{7}$.

$\therefore$ slope of the given line $= \frac{1}{7}$.

$\therefore$ slope of a line perpendicular to the given line $= -7 \left[\because m_2 = \frac{-1}{m_1} \right]$.

Thus, $m = -7$ and $d = 3$, where $d = x$ -intercept.

Now, the equation of a line with slope -7 and x -intercept 3 is given by

$$y = -7(x - 3) \quad [\because y = m(x - d)]$$

$\Rightarrow 7x + y - 21 = 0$, which is the required equation.

EXAMPLE 17 Find the equations of the lines through the point $(3, 2)$, which make an angle of 45° with the line $x - 2y = 3$.

SOLUTION Let the slope of the required line be m .

Then, its equation is $\frac{y - 2}{x - 3} = m$ (i)

Given line is $x - 2y = 3 \Rightarrow y = \frac{1}{2}x - \frac{3}{2}$ (ii)

Clearly, the slope of this line is $\frac{1}{2}$.

It is given that the angle between (i) and (ii) is 45° .

$$\therefore \left| \frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} \right| = \tan 45^\circ \quad \left[\text{using } \tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \right]$$

$$\Leftrightarrow \frac{2m - 1}{2 + m} = 1 \quad \text{or} \quad \frac{2m - 1}{2 + m} = -1 \Leftrightarrow m = 3 \quad \text{or} \quad m = \frac{-1}{3}$$

$\therefore$ the required equation is $\frac{y - 2}{x - 3} = 3$ or $\frac{y - 2}{x - 3} = \frac{-1}{3}$

i.e., $3x - y - 7 = 0$ or $x + 3y - 9 = 0$.



EXERCISE 20D

- Find the equation of the line whose
 - slope = 3 and y -intercept = 5
 - slope = -1 and y -intercept = 4
 - slope = $-\frac{2}{5}$ and y -intercept = -3
- Find the equation of the line which makes an angle of 30° with the positive direction of the x -axis and cuts off an intercept of 4 units with the negative direction of the y -axis.
- Find the equation of the line whose inclination is $\frac{5\pi}{6}$ and which makes an intercept of 6 units on the negative direction of the y -axis.
- Find the equation of the line cutting off an intercept -2 from the y -axis and equally inclined to the axes.
- Find the equation of the bisectors of the angles between the coordinate axes.
- Find the equation of the line through the point $(-1, 5)$ and making an intercept of -2 on the y -axis.
- Find the equation of the line which is parallel to the line $2x - 3y = 8$ and whose y -intercept is 5 units.
- Find the equation of the line passing through the point $(0, 3)$ and perpendicular to the line $x - 2y + 5 = 0$.
- Find the equation of the line passing through the point $(2, 3)$ and perpendicular to the line $4x + 3y = 10$.
- Find the equation of the line passing through the point $(2, 4)$ and perpendicular to the x -axis.
- Find the equation of the line that has x -intercept -3 and which is perpendicular to the line $3x + 5y = 4$.
- Find the equation of the line which is perpendicular to the line $3x + 2y = 8$ and passes through the midpoint of the line joining the points $(6, 4)$ and $(4, -2)$.
- Find the equation of the line whose y -intercept is -3 and which is perpendicular to the line joining the points $(-2, 3)$ and $(4, -5)$.
- Find the equation of the line passing through $(-3, 5)$ and perpendicular to the line through the points $(2, 5)$ and $(-3, 6)$.
- A line perpendicular to the line segment joining the points $(1, 0)$ and $(2, 3)$ divides it in the ratio $1 : 2$. Find the equation of the line.

ANSWERS (EXERCISE 20D)

- (i) $3x - y + 5 = 0$ (ii) $x + y - 4 = 0$ (iii) $2x + 5y + 15 = 0$
- $x - \sqrt{3}y - 4\sqrt{3} = 0$
- $x + \sqrt{3}y + 6\sqrt{3} = 0$
- $x - y - 2 = 0$ or $x + y + 2 = 0$
- $x - y = 0$ or $x + y = 0$



6. $7x + y + 2 = 0$ 7. $2x - 3y + 15 = 0$ 8. $2x + y - 3 = 0$ 9. $3x - 4y + 6 = 0$
 10. $x = 2$ 11. $5x - 3y + 15 = 0$ 12. $2x - 3y - 7 = 0$ 13. $3x - 4y - 12 = 0$
 14. $5x - y + 20 = 0$ 15. $3x + 9y - 13 = 0$

HINTS TO SOME SELECTED QUESTIONS

2. $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$ and $c = -4$.
3. $m = \tan \frac{5\pi}{6} = \tan \left(\pi - \frac{\pi}{6} \right) = -\tan \frac{\pi}{6} = \frac{-1}{\sqrt{3}}$. And, $c = -6$.
4. $m = \tan 45^\circ = 1$ or $m = \tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1$. Also, $c = -2$.
5. $m = \tan 45^\circ = 1$ or $m = \tan 135^\circ = -1$. And, $c = 0$.
6. Let the required equation be $y = mx - 2$.
 Since it passes through $(-1, 5)$, we have $m \times (-1) - 2 = 5 \Rightarrow m = -7$.
7. Given line is $y = \frac{2}{3}x - \frac{8}{3}$. Its slope $= \frac{2}{3}$.
 $\therefore$ slope of the required line $= \frac{2}{3}$ and $c = 5$.
8. Given line is $y = \frac{1}{2}x + \frac{5}{2}$. Its slope is $\frac{1}{2}$.
 $m \times \frac{1}{2} = -1 \Rightarrow m = -2$.
 Let $y = -2x + c$ be the required line.
 Since it passes through $(0, 3)$, we have $3 = (-2) \times 0 + c \Rightarrow c = 3$.
 $\therefore m = -2$ and $c = 3$.
10. Required line is parallel to the y -axis. So, its equation is $x = 2$.
11. Required line passes through $(-3, 0)$.
12. Required line passes through the point $\left(\frac{5+3}{2}, 0 \right)$, i.e., $(4, 0)$.
13. Let $A(-2, 3)$ and $B(4, -5)$ be the given points.
 Slope of $AB = \frac{-5-3}{4+2} = \frac{-4}{3}$. So, $m \times \left(\frac{-4}{3} \right) = -1 \Rightarrow m = \frac{3}{4}$.
 $\therefore y = \frac{3}{4}x - 3$ [$\because c = -3$].
14. Given points are $A(2, 5)$ and $B(-3, 6)$.
 Slope of $AB = \frac{6-5}{-3-2} = \frac{-1}{5}$. So, $m \times \left(\frac{-1}{5} \right) = -1 \Rightarrow m = 5$.
 $\therefore$ required equation is $\frac{y-5}{x+3} = 5 \Leftrightarrow 5x - y + 20 = 0$.



15. Let M divide the join of $A(1, 0)$ and $B(2, 3)$ in the ratio $1 : 2$.

Then, the point M is $\left(\frac{1 \times 2 + 2 \times 1}{1 + 2}, \frac{1 \times 3 + 2 \times 0}{1 + 2}\right)$, i.e., $M\left(\frac{4}{3}, 1\right)$.

$$\text{Slope of } AB = \frac{3 - 0}{2 - 1} = 3.$$

Let L be the required line. Then, slope of $L = -\frac{1}{3}$ and it passes through $M\left(\frac{4}{3}, 1\right)$.

Let L be the required line. Then, $L \perp AB$.

$\therefore$ slope of $L = -\frac{1}{3}$ and it passes through $M\left(\frac{4}{3}, 1\right)$.

EQUATION OF A LINE IN INTERCEPTS FORM

THEOREM 1 Prove that the equation of a line making intercepts a and b on the x -axis and the y -axis respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

PROOF Let $X'OX$ and YOY' be the coordinate axes.

Let AB be the line cutting the x -axis and the y -axis at A and B respectively such that $OA = a$ and $OB = b$.

Then, these points are $A(a, 0)$ and $B(0, b)$.

So, the equation of line AB is given by

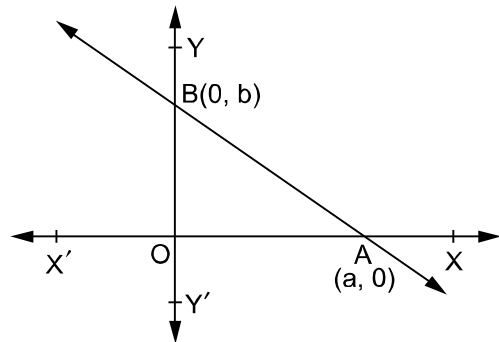
$$\frac{y - 0}{x - a} = \frac{b - 0}{0 - a} \quad [\text{two-point form}]$$

$$\Leftrightarrow \frac{y}{x - a} = \frac{b}{-a} \Leftrightarrow b(x - a) = -ay$$

$$\Leftrightarrow bx + ay = ab$$

$$\Leftrightarrow \frac{x}{a} + \frac{y}{b} = 1 \quad [\text{on dividing both sides by } ab].$$

Thus, the line making intercepts a and b on the x - and y -axes respectively is $\frac{x}{a} + \frac{y}{b} = 1$.



SOLVED EXAMPLES

EXAMPLE 1 Find the equation of the line which makes intercepts 2 and -3 on the x -axis and the y -axis respectively.

SOLUTION We know that the equation of a line making intercepts a and b on the x -axis and y -axis respectively, is $\frac{x}{a} + \frac{y}{b} = 1$.



Here $a = 2$ and $b = -3$.

Hence, the required equation is

$$\frac{x}{2} + \frac{y}{-3} = 1 \Leftrightarrow \frac{x}{2} - \frac{y}{3} = 1 \Leftrightarrow 3x - 2y - 6 = 0.$$

EXAMPLE 2 Find the equation of the line which passes through the point $(3, 4)$ and the sum of whose intercepts on the axes is 14.

SOLUTION Let the intercepts made by the line on the x -axis and y -axis be a and $(14 - a)$ respectively.

Then, its equation is $\frac{x}{a} + \frac{y}{(14 - a)} = 1$ [intercept form].

Since it passes through the point $(3, 4)$, we have

$$\begin{aligned} \frac{3}{a} + \frac{4}{(14 - a)} = 1 &\Leftrightarrow 3(14 - a) + 4a = a(14 - a) \\ &\Leftrightarrow a^2 - 13a + 42 = 0 \Leftrightarrow (a - 6)(a - 7) = 0 \\ &\Leftrightarrow a = 6 \text{ or } a = 7. \end{aligned}$$

Now, $a = 6 \Leftrightarrow b = 14 - 6 = 8$.

And, $a = 7 \Leftrightarrow b = 14 - 7 = 7$.

So, the required equation is

$$\frac{x}{6} + \frac{y}{8} = 1 \text{ or } \frac{x}{7} + \frac{y}{7} = 1$$

i.e., $4x + 3y - 24 = 0$ or $x + y - 7 = 0$.

EXAMPLE 3 Find the equations of the lines which cut off intercepts on the axes whose sum and product are 1 and -6 respectively.

SOLUTION Let the required equation be $\frac{x}{a} + \frac{y}{b} = 1$.

Then x -intercept $= a$ and y -intercept $= b$.

$$\therefore a + b = 1 \quad \dots (i)$$

$$\text{and } ab = -6 \quad \dots (ii)$$

Putting $b = \frac{-6}{a}$ from (ii) in (i), we get

$$\begin{aligned} a - \frac{6}{a} = 1 &\Leftrightarrow a^2 - a - 6 = 0 \Leftrightarrow (a - 3)(a + 2) = 0 \\ &\Leftrightarrow a = 3 \text{ or } a = -2. \end{aligned}$$

Now, $a = 3 \Leftrightarrow b = (1 - a) = (1 - 3) = -2$.

And, $a = -2 \Leftrightarrow b = (1 - a) = (1 + 2) = 3$.

$\therefore$ the required equation is

$$\frac{x}{3} + \frac{y}{-2} = 1 \text{ or } \frac{x}{-2} + \frac{y}{3} = 1$$

i.e., $2x - 3y - 6 = 0$ or $3x - 2y + 6 = 0$.

EXAMPLE 4 If $M(a, b)$ is the midpoint of a line segment intercepted between the axes, show that the equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$.



SOLUTION Let the required equation of the line be

$$\frac{x}{c} + \frac{y}{d} = 1. \quad \dots (i)$$

Then, it cuts the x -axis and y -axis at the points $A(c, 0)$ and $B(0, d)$.

Let $M(a, b)$ be the midpoint of AB .

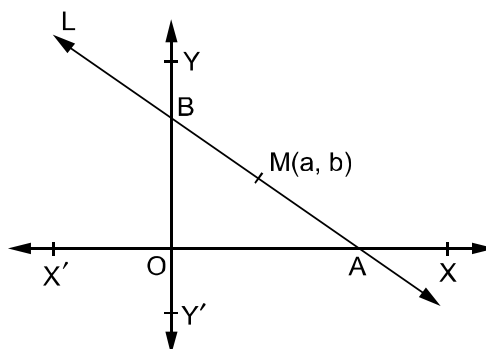
$$\text{Then, } \frac{c+0}{2} = a \text{ and } \frac{0+d}{2} = b$$

$$\Leftrightarrow c = 2a \text{ and } d = 2b.$$

So, the required equation is

$$\frac{x}{2a} + \frac{y}{2b} = 1 \Leftrightarrow \frac{x}{a} + \frac{y}{b} = 2.$$

Hence, the required equation is $\frac{x}{a} + \frac{y}{b} = 2$.



EXAMPLE 5 Find the equation of a line which passes through the point $(-3, 7)$ and makes intercepts on the axes, equal in magnitude but opposite in sign.

SOLUTION Let the required line make intercepts a and $-a$ on the x -axis and y -axis respectively.

$$\text{Then, its equation is } \frac{x}{a} + \frac{y}{-a} = 1 \Leftrightarrow x - y = a. \quad \dots (i)$$

Since this line passes through the point $(-3, 7)$, we have

$$a = (-3 - 7) = -10.$$

So, the required equation of the line is

$$\frac{x}{-10} + \frac{y}{10} = 1 \Leftrightarrow -x + y = 10 \Leftrightarrow x - y + 10 = 0.$$

Hence, the required equation is $x - y + 10 = 0$.

EXAMPLE 6 Find the intercepts cut off by the line $2x - y + 16 = 0$ on the coordinate axes.

SOLUTION We have

$$2x - y + 16 = 0 \Leftrightarrow 2x - y = -16 \Leftrightarrow \frac{x}{-8} + \frac{y}{16} = 1$$

[on dividing both sides by -16]

$\therefore$ the x -intercept $= -8$ and the y -intercept $= 16$.

EXAMPLE 7 Find the equation of the line so that the line segment intercepted between the axes is bisected at the point $(2, 3)$.

SOLUTION Let AB be the given line segment making intercepts a and b on the x -axis and y -axis respectively.

Then, the equation of line AB is $\frac{x}{a} + \frac{y}{b} = 1$.



Moreover, these points are $A(a, 0)$ and $B(0, b)$. Let $M(2, 3)$ be the midpoint of AB . Then,

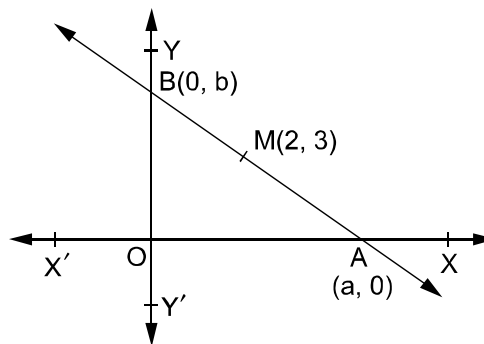
$$\frac{a+0}{2} = 2 \quad \text{and} \quad \frac{0+b}{2} = 3$$

$$\Rightarrow a = 4 \text{ and } b = 6.$$

Hence, the required equation of the given line is

$$\frac{x}{4} + \frac{y}{6} = 1 \Leftrightarrow 3x + 2y - 12 = 0.$$

Hence, the required equation is $3x + 2y - 12 = 0$.



EXAMPLE 8 Find the equation of the line so that the line segment intercepted between the axes is divided by the point $P(-5, 4)$ in the ratio $1 : 2$.

SOLUTION Let AB be the given line segment making intercepts a and b on the x -axis and y -axis respectively.

Then, the equation of line AB is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

So, these points are $A(a, 0)$ and $B(0, b)$.

Now, $P(-5, 4)$ divides AB in the ratio $1 : 2$.

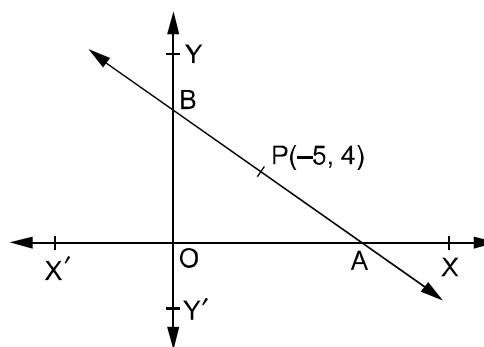
$$\therefore \frac{(1 \times 0 + 2 \times a)}{(1 + 2)} = -5 \text{ and } \frac{(1 \times b + 2 \times 0)}{(1 + 2)} = 4 \Leftrightarrow a = \frac{-15}{2} \text{ and } b = 12.$$

So, the required equation of the line is

$$\frac{x}{\left(\frac{-15}{2}\right)} + \frac{y}{12} = 1 \Leftrightarrow \frac{-2x}{15} + \frac{y}{12} = 1$$

$$\Leftrightarrow 8x - 5y + 60 = 0.$$

Hence, the required equation of the line is $8x - 5y + 60 = 0$.



EXAMPLE 9 Find the equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point, where it meets the y -axis.

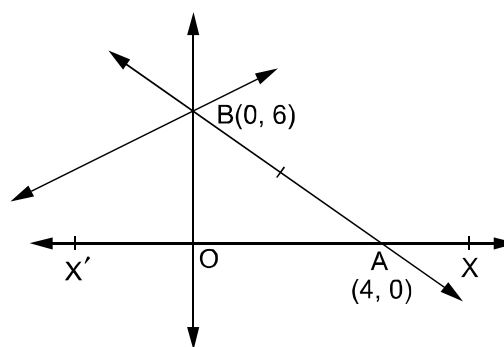
SOLUTION The given line is $\frac{x}{4} + \frac{y}{6} = 1$.

It meets the y -axis at $B(0, 6)$.

$$\text{Also, } \frac{x}{4} + \frac{y}{6} = 1 \Leftrightarrow 3x + 2y = 12$$

$$\Leftrightarrow y = \frac{-3}{2}x + 6.$$

Slope of the given line is $\frac{-3}{2}$.



Let the slope of the line perpendicular to the given line be m . Then,

$$m \times \left(-\frac{3}{2}\right) = -1 \Rightarrow m = \frac{2}{3} \quad [\because m_1 m_2 = -1].$$

Now, the required line has slope $\frac{2}{3}$ and passes through the point $B(0, 6)$. So, its equation is

$$\frac{y - 6}{x - 0} = \frac{2}{3} \Leftrightarrow 2x = 3y - 18 \Leftrightarrow 2x - 3y + 18 = 0.$$

Hence, the required equation is $2x - 3y + 18 = 0$.

EXAMPLE 10 Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$, which has equal intercepts on the axes.

AN IMPORTANT NOTE The point of intersection of two lines is obtained by solving their equations simultaneously.

SOLUTION Let the equation of the required line be $\frac{x}{a} + \frac{y}{a} = 1$,
i.e., $x + y = a$... (i)

The equations of the given lines are:

$$4x + 7y = 3 \quad \dots \text{(ii)}$$

$$2x - 3y = -1 \quad \dots \text{(iii)}$$

On solving (ii) and (iii), we get $x = \frac{1}{13}$, $y = \frac{5}{13}$.

So, the point of intersection of the given lines is $P\left(\frac{1}{13}, \frac{5}{13}\right)$.

Now, line (i) passes through the point $P\left(\frac{1}{13}, \frac{5}{13}\right)$.

Putting $x = \frac{1}{13}$ and $y = \frac{5}{13}$ in (i), we get

$$a = \left(\frac{1}{13} + \frac{5}{13}\right) = \frac{6}{13}.$$

Hence, the required equation is $x + y = \frac{6}{13}$,

i.e., $13x + 13y - 6 = 0$.

EXAMPLE 11 Find the area of the triangle formed by the coordinate axes and the line $ax + by = 2ab$.

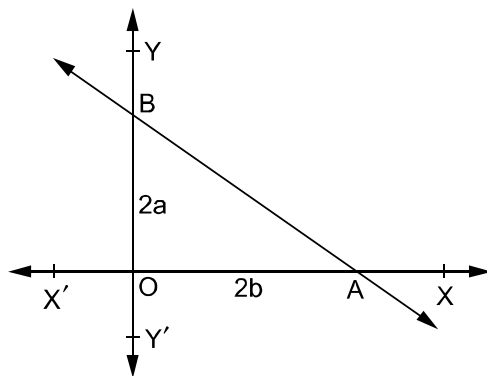
SOLUTION Let $X'OX$ and YOY' be the coordinate axes.

Now, $ax + by = 2ab \Rightarrow \frac{x}{2b} + \frac{y}{2a} = 1$.

Suppose that this line meets the x -axis and y -axis at A and B respectively.

Then, $OA = 2b$ and $OB = 2a$.





$$\begin{aligned}\therefore \text{ area of } \triangle BOA &= \left(\frac{1}{2} \times OA \times OB \right) \text{ sq units} \\ &= \left(\frac{1}{2} \times 2b \times 2a \right) = 2ab \text{ sq units.}\end{aligned}$$

EXAMPLE 12 The area of the triangle formed by the coordinate axes and a line is 6 square units and the length of its hypotenuse is 5 units. Find the equation of the line.

SOLUTION Let $X'OX$ and $Y'OY'$ be the coordinate axes, and let AB be the part of the given line intercepted by the axes.

Let $OA = a$ and $OB = b$.

$$\text{Then, } \frac{1}{2}ab = 6 \Leftrightarrow ab = 12 \quad \dots (i)$$

$$\text{and, } a^2 + b^2 = 25 \quad \dots (ii)$$

Putting $b = \frac{12}{a}$ from (i) in (ii), we get

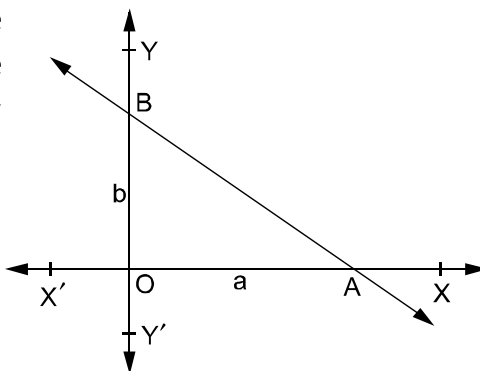
$$\begin{aligned}a^2 + \frac{144}{a^2} &= 25 \Rightarrow a^4 - 25a^2 + 144 = 0 \\ &\Rightarrow (a^2 - 16)(a^2 - 9) = 0 \Rightarrow a^2 = 16 \text{ or } a^2 = 9 \\ &\Rightarrow a = 4 \text{ or } a = 3 \quad [\because \text{length is always positive}].\end{aligned}$$

Now, $(a = 4 \Rightarrow b = 3)$ and $(a = 3 \Rightarrow b = 4)$.

Hence, the required equation of the line is

$$\frac{x}{4} + \frac{y}{3} = 1 \quad \text{or} \quad \frac{x}{3} + \frac{y}{4} = 1$$

$$\text{i.e., } 3x + 4y - 12 = 0 \quad \text{or} \quad 4x + 3y - 12 = 0.$$



EXERCISE 20E

- Find the equation of the line which cuts off intercepts -3 and 5 on the x -axis and y -axis respectively.
- Find the equation of the line which cuts off intercepts 4 and -6 on the x -axis and y -axis respectively.



3. Find the equation of the line that cuts off equal intercepts on the coordinate axes and passes through the point (4, 7).
4. Find the equation of the line which passes through the point (3, -5) and cuts off intercepts on the axes which are equal in magnitude but opposite in sign.
5. Find the equation of the line passing through the point (2, 2) and cutting off intercepts on the axes, whose sum is 9.
6. Find the equation of the line which passes through the point (22, -6) and whose intercept on the x -axis exceeds the intercept on the y -axis by 5.
7. Find the equation of the line whose portion intercepted between the axes is bisected at the point (3, -2).
8. Find the equation of the line whose portion intercepted between the coordinate axes is divided at the point (5, 6) in the ratio 3 : 1.
9. A straight line passes through the point (-5, 2) and the portion of the line intercepted between the axes is divided at this point in the ratio 2 : 3. Find the equation of the line.
10. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points (8, -9) and (12, -15), find the values of a and b .

ANSWERS (EXERCISE 20E)

1. $5x - 3y + 15 = 0$ 2. $3x - 2y - 12 = 0$ 3. $x + y - 11 = 0$ 4. $x - y - 8 = 0$
 5. $x + 2y - 6 = 0$ or $2x + y - 6 = 0$ 6. $6x + 11y - 66 = 0$ or $x + 2y - 10 = 0$
 7. $2x - 3y - 12 = 0$ 8. $2x + 5y - 40 = 0$ 9. $3x - 5y + 25 = 0$ 10. $a = 2, b = 3$

HINTS TO SOME SELECTED QUESTIONS

6. Let the required equation be $\frac{x}{a+5} + \frac{y}{a} = 1$.

Since it passes through (22, -6), we have $\frac{22}{a+b} - \frac{6}{a} = 1$.

This gives $a^2 - 11a + 30 = 0 \Leftrightarrow (a-6)(a-5) = 0 \Leftrightarrow a = 6$ or $a = 5$.

So, the required equations are $\frac{x}{11} + \frac{y}{6} = 1$ or $\frac{x}{10} + \frac{y}{5} = 1$.

7. Let the required equation be $\frac{x}{a} + \frac{y}{b} = 1$.

Then, it cuts the axes at $A(a, 0)$ and $B(0, b)$.

$$\therefore \left(\frac{a+0}{2} = 3 \text{ and } \frac{0+b}{2} = -2 \right) \Leftrightarrow a = 6, b = -4.$$

8. Let the required equation be $\frac{x}{a} + \frac{y}{b} = 1$.

Then, it cuts the axes at $A(a, 0)$ and $B(0, b)$.

Let $P(5, 6)$ divide AB in the ratio 3 : 1.



Then, $\frac{3 \times 0 + 1 \times a}{3 + 1} = 5$ and $\frac{3 \times b + 1 \times 0}{3 + 1} = 6 \Leftrightarrow a = 20$ and $b = 8$.

10. Since $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points $A(8, -9)$ and $B(12, -15)$,

we have $\frac{8}{a} - \frac{9}{b} = 1$ and $\frac{12}{a} - \frac{15}{b} = 1$.

On solving, we get $\frac{3}{b} = 1$ and $\frac{8}{a} = 4$. This gives $a = 2, b = 3$.

EQUATION OF A LINE IN NORMAL FORM

THEOREM 1 Let p be the length of perpendicular (or normal) from the origin to a given non-vertical line L , and let α be the angle between the normal and the positive direction of the x -axis. Then, prove that the equation of the line L is given by

$$x \cos \alpha + y \sin \alpha = p.$$

PROOF Let $X'OX$ and YOY' be the coordinate axes and let L be the given line.

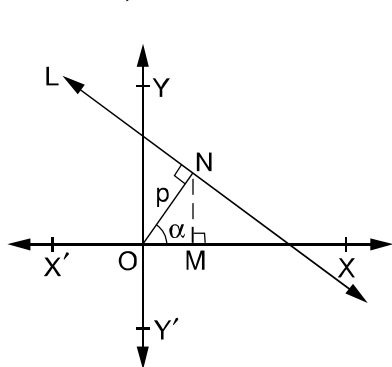
Draw ON perpendicular to L .

Let $ON = p$, and let the angle between the positive direction of the x -axis and ON be α .

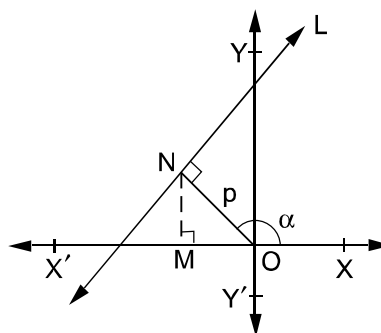
Draw perpendicular NM on the x -axis.

All possible positions of the line L are shown below.

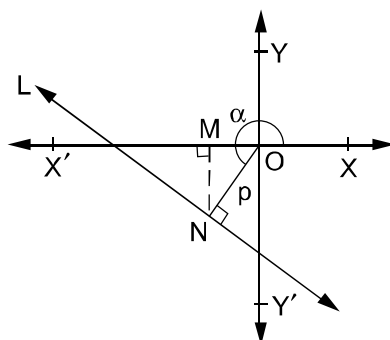
In each case, we have



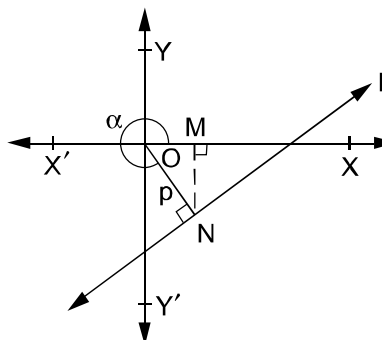
(i)



(ii)



(iii)



(iv)



$$\frac{OM}{ON} = \cos \alpha \quad \text{and} \quad \frac{NM}{ON} = \sin \alpha$$

$$\Leftrightarrow OM = p \cos \alpha \quad \text{and} \quad NM = p \sin \alpha.$$

Thus, the coordinates of N are $(p \cos \alpha, p \sin \alpha)$.

Moreover, line L is perpendicular to ON .

$$\therefore \text{ slope of the line } L = \frac{-1}{\text{slope of } ON} = \frac{-1}{\tan \alpha} = \frac{-\cos \alpha}{\sin \alpha}.$$

Thus, the line L passes through the point $N(p \cos \alpha, p \sin \alpha)$ and has a slope $m = \frac{-\cos \alpha}{\sin \alpha}$.

So, the equation of the line L is given by

$$\frac{y - p \sin \alpha}{x - p \cos \alpha} = \frac{-\cos \alpha}{\sin \alpha} \quad [\text{point-slope form}]$$

$$\Leftrightarrow (y - p \sin \alpha) \sin \alpha = -\cos \alpha (x - p \cos \alpha)$$

$$\Leftrightarrow x \cos \alpha + y \sin \alpha = p(\cos^2 \alpha + \sin^2 \alpha)$$

$$\Leftrightarrow x \cos \alpha + y \sin \alpha = p, \text{ which is the required equation.}$$

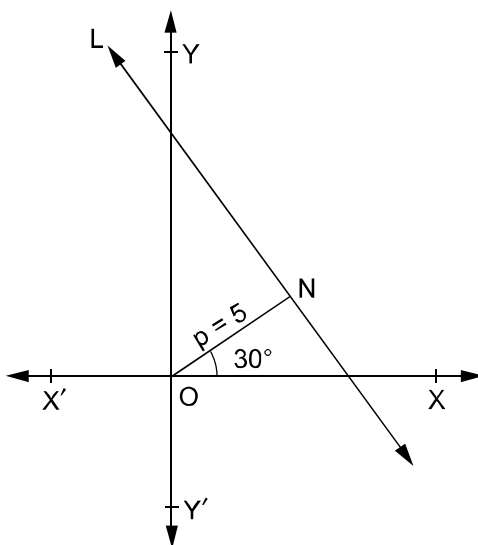
SOLVED EXAMPLES

EXAMPLE 1 Find the equation of a line whose perpendicular distance from the origin is 5 units and the angle between the positive direction of the x -axis and the perpendicular is 30° .

SOLUTION Here $p = 5$ units and $\alpha = 30^\circ$.

So, the equation of the given line in normal form is

$$x \cos \alpha + y \sin \alpha = p, \text{ where } \alpha = 30^\circ \text{ and } p = 5 \text{ units}$$



$$\Leftrightarrow x \cos 30^\circ + y \sin 30^\circ = 5$$



$$\Leftrightarrow x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) = 5$$

$$\Leftrightarrow \sqrt{3}x + y - 10 = 0,$$

which is the required equation.

EXAMPLE 2 Find the equation of the line whose perpendicular distance from the origin is 3 units and the angle between the positive direction of x -axis and the perpendicular is 15° .

SOLUTION Here $p = 3$ units and $\alpha = 15^\circ$.

So, the equation of the given line in normal form is

$x \cos \alpha + y \sin \alpha = p$, where $\alpha = 15^\circ$ and $p = 3$ units

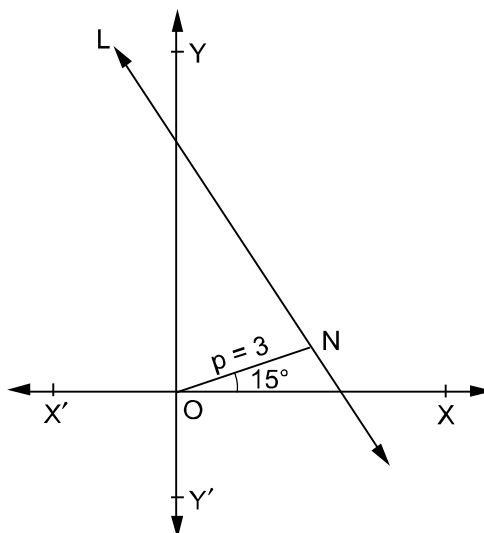
$$\Leftrightarrow x \cos 15^\circ + y \sin 15^\circ = 3$$

$$\Leftrightarrow \frac{x(\sqrt{3} + 1)}{2\sqrt{2}} + \frac{y(\sqrt{3} - 1)}{2\sqrt{2}} = 3$$

$$[\because \cos 15^\circ = \cos (45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$\sin 15^\circ = \sin (45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ]$$

$$\Leftrightarrow (\sqrt{3} + 1)x + (\sqrt{3} - 1)y - 6\sqrt{2} = 0, \text{ which is the required equation.}$$



EXAMPLE 3 Find the equation of a line whose perpendicular distance from the origin is $\sqrt{8}$ units and the angle between the positive direction of the x -axis and the perpendicular is 135° .

SOLUTION Here $p = \sqrt{8}$ units and $\alpha = 135^\circ$.

So, the equation of the given line in normal form is

$$x \cos \alpha + y \sin \alpha = p,$$

where $\alpha = 135^\circ$ and $p = \sqrt{8}$ units

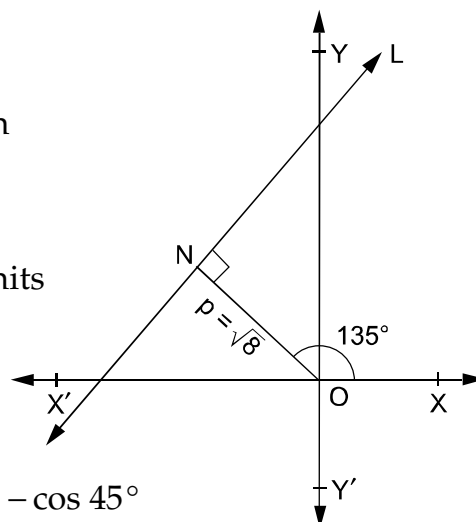
$$\Leftrightarrow x \cos 135^\circ + y \sin 135^\circ = \sqrt{8}$$

$$\Leftrightarrow x\left(\frac{-1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) = \sqrt{18}$$

$$[\because \cos 135^\circ = \cos (180^\circ - 45^\circ) = -\cos 45^\circ$$

$$\sin 135^\circ = \sin (180^\circ - 45^\circ) = \sin 45^\circ]$$

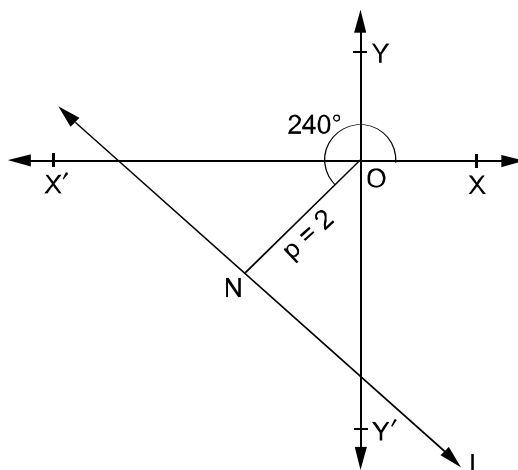
$$\Leftrightarrow -x + y = 4 \Leftrightarrow x - y + 4 = 0, \text{ which is the required equation.}$$



EXAMPLE 4 Find the equation of a line whose perpendicular distance from the origin is 2 units and the angle between the perpendicular segment and the positive direction of the x -axis is 240° .



SOLUTION Here, $p = 2$ units and $\alpha = 240^\circ$.



So, the equation of the given line in normal form is

$$x \cos \alpha + y \sin \alpha = p, \text{ where } \alpha = 240^\circ \text{ and } p = 2 \text{ units}$$

$$\Leftrightarrow x \cos 240^\circ + y \sin 240^\circ = 2$$

$$\Leftrightarrow x \left(\frac{-1}{2} \right) + y \left(\frac{-\sqrt{3}}{2} \right) = 2 \quad [\because \cos 240^\circ = \cos (180^\circ + 60^\circ) = -\cos 60^\circ \\ \sin 240^\circ = \sin (180^\circ + 60^\circ) = -\sin 60^\circ]$$

$$\Leftrightarrow -x - \sqrt{3}y = 4 \Leftrightarrow x + \sqrt{3}y + 4 = 0, \text{ which is the required equation.}$$

EXERCISE 20F

1. Find the equation of the line for which

(i) $p = 3$ and $\alpha = 45^\circ$

(ii) $p = 5$ and $\alpha = 135^\circ$

(iii) $p = 8$ and $\alpha = 150^\circ$

(iv) $p = 3$ and $\alpha = 225^\circ$

(v) $p = 2$ and $\alpha = 300^\circ$

(vi) $p = 4$ and $\alpha = 180^\circ$

2. The length of the perpendicular segment from the origin to a line is 2 units and the inclination of this perpendicular is α such that $\sin \alpha = \frac{1}{3}$ and α is acute. Find the equation of the line.

3. Find the equation of the line which is at a distance of 3 units from the origin such that $\tan \alpha = \frac{5}{12}$, where α is the acute angle which this perpendicular makes with the positive direction of the x -axis.

ANSWERS (EXERCISE 20F)

1. (i) $x + y - 3\sqrt{2} = 0$ (ii) $x - y + 5\sqrt{2} = 0$ (iii) $\sqrt{3}x - y + 16 = 0$

(iv) $x + y + 3\sqrt{2} = 0$ (v) $x - \sqrt{3}y - 4 = 0$ (vi) $x + 4 = 0$

2. $2\sqrt{2}x + y - 6 = 0$

3. $12x + 5y - 39 = 0$



GENERAL EQUATION OF A LINE

THEOREM 1 *Prove that the equation $Ax + By + C = 0$ always represents a line, provided A and B are not simultaneously zero.*

PROOF Let $Ax + By + C = 0$, where $A \neq 0$ or $B \neq 0$.

CASE 1 When $A = 0$ and $B \neq 0$

In this case, the given equation becomes

$$By + C = 0 \Rightarrow y = \left(\frac{-C}{B} \right),$$

which represents a line parallel to the x -axis.

CASE 2 When $A \neq 0$ and $B = 0$

In this case, the given equation becomes

$$Ax + C = 0 \Rightarrow x = \left(\frac{-C}{A} \right),$$

which represents a line parallel to the y -axis.

CASE 3 When $A \neq 0$ and $B \neq 0$

In this case, we have

$$\begin{aligned} Ax + By + C = 0 &\Rightarrow By = -Ax + (-C) \\ &\Rightarrow y = \left(\frac{-A}{B} \right) x + \left(\frac{-C}{B} \right). \end{aligned}$$

This is clearly an equation of a line in slope-intercept form,

$$y = mx + c, \text{ where } m = \left(\frac{-A}{B} \right) \text{ and } c = \frac{-C}{B}.$$

Thus, $Ax + By + C = 0$, when $A \neq 0$ or $B \neq 0$, always represents a line.

THEOREM 2 *Prove that every line has an equation of the form $Ax + By + c = 0$, where A, B, C are constants.*

PROOF For a given line, there are two possibilities.

CASE 1 The given line is parallel to the y -axis.

In this case, the equation is of the form, $x = c$.

$$\begin{aligned} \text{Now, } x = c &\Leftrightarrow 1 \cdot x + 0 \cdot y - c = 0 \\ &\Leftrightarrow Ax + By + C = 0, \text{ where } C = -c. \end{aligned}$$

Thus, it is of the form $Ax + By + C = 0$.

CASE 2 The given line is not parallel to the y -axis.

In this case, the equation is of the form

$$\begin{aligned} y = mx + c &\Rightarrow mx - y + c = 0 \\ &\Rightarrow Ax + By + C = 0, \end{aligned}$$

where $A = m$, $B = -1$ and $C = c$.

Thus, it takes the form $Ax + By + C = 0$.

REDUCTION OF GENERAL FORM TO STANDARD FORM

Let the given equation of the line be $Ax + By + C = 0$.



I. SLOPE-INTERCEPT FORM:

$$Ax + By + C = 0 \Rightarrow By = -Ax - C$$

$$\Rightarrow y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right).$$

$$\Rightarrow y = mx + c, \text{ where } m = \frac{-A}{B} \text{ and } c = \frac{-C}{B}.$$

This is clearly the equation of a line in *slope-intercept* form, $y = mx + c$,

$$\text{where } m = \frac{-A}{B} \text{ and } c = \frac{-C}{B}.$$

II. INTERCEPTS FORM:

$$Ax + By + C = 0 \Rightarrow Ax + By = -C$$

$$\Rightarrow \left(\frac{A}{-C}\right)x + \left(\frac{B}{-C}\right)y = 1 \quad [\text{on dividing both sides by } -C]$$

$$\Rightarrow \frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1, \text{ where } a = \frac{-C}{A} \text{ and } b = \frac{-C}{B}.$$

This is clearly the equation of a line in intercept form, $\frac{x}{a} + \frac{y}{b} = 1$,

$$\text{where } a = \frac{-C}{A} \text{ and } b = \frac{-C}{B}.$$

III. NORMAL FORM:

$$Ax + By + C = 0 \Rightarrow \frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

[on dividing throughout by $\sqrt{A^2 + B^2}$]

$$\Rightarrow \frac{-A}{\sqrt{A^2 + B^2}}x + \frac{-B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}$$

$$\Rightarrow x \cos \alpha + y \sin \alpha = p,$$

$$\text{where } \cos \alpha = \frac{-A}{\sqrt{A^2 + B^2}}, \sin \alpha = \frac{-B}{\sqrt{A^2 + B^2}} \text{ and } p = \frac{C}{\sqrt{A^2 + B^2}}.$$

This is the equation of a line in normal form, $x \cos \alpha + y \sin \alpha = p$,

$$\text{where } \cos \alpha = \frac{-A}{\sqrt{A^2 + B^2}}, \sin \alpha = \frac{-B}{\sqrt{A^2 + B^2}} \text{ and } p = \frac{C}{\sqrt{A^2 + B^2}}.$$

SOLVED EXAMPLES

EXAMPLE 1 Reduce the equation $\sqrt{3}x + y + 2 = 0$ to

(i) slope-intercept form and find the slope and y-intercept.

(ii) intercepts form and find the intercepts on the axes.



SOLUTION We have

$$(i) \sqrt{3}x + y + 2 = 0 \Rightarrow y = -\sqrt{3}x - 2.$$

This is of the form $y = mx + c$, where $m = -\sqrt{3}$ and $c = -2$.

$\therefore y = -\sqrt{3}x - 2$ is in slope-intercept form.

Here, slope $= -\sqrt{3}$ and y -intercept $= -2$.

$$\begin{aligned} (ii) \sqrt{3}x + y + 2 = 0 &\Rightarrow \sqrt{3}x + y = -2 \\ &\Rightarrow \left(\frac{\sqrt{3}}{-2}\right)x + \left(\frac{1}{-2}\right)y = 1 \\ &\Rightarrow \frac{x}{\left(\frac{-2}{\sqrt{3}}\right)} + \frac{y}{-2} = 1. \end{aligned}$$

This is of the form $\frac{x}{a} + \frac{y}{b} = 1$, where $a = \frac{-2}{\sqrt{3}}$ and $b = -2$.

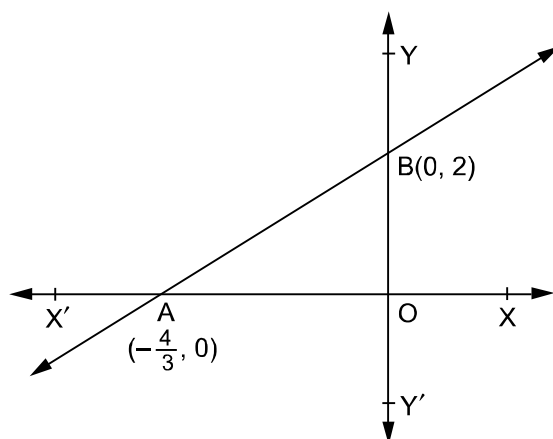
Thus, $\frac{x}{\left(\frac{-2}{\sqrt{3}}\right)} + \frac{y}{-2} = 1$ is in intercepts form.

Here, x -intercept $= \frac{-2}{\sqrt{3}}$ and y -intercept $= -2$.

EXAMPLE 2 Reduce the equation $3x - 2y + 4 = 0$ to intercepts form and find the length of the segment intercepted between the axes.

SOLUTION We have

$$\begin{aligned} 3x - 2y + 4 = 0 &\Rightarrow 3x - 2y = -4 \\ &\Rightarrow \frac{3x}{-4} + \frac{y}{2} = 1 \quad [\text{on dividing both sides by } -4] \\ &\Rightarrow \frac{x}{\left(\frac{-4}{3}\right)} + \frac{y}{2} = 1 \\ &\Rightarrow \frac{x}{a} + \frac{y}{b} = 1, \text{ where } a = \frac{-4}{3} \text{ and } b = 2. \end{aligned}$$



$$\therefore \frac{x}{\left(\frac{-4}{3}\right)} + \frac{y}{2} = 1 \text{ is the required equation in intercepts form.}$$

$$x\text{-intercept} = \frac{-4}{3} \text{ and } y\text{-intercept} = 2.$$

If AB is the part of the line intercepted between the axes then the end points of this line segment are $A\left(\frac{-4}{3}, 0\right)$ and $B(0, 2)$.

$$\text{Length } AB = \sqrt{\left(\frac{-4}{3} - 0\right)^2 + (0 - 2)^2} = \sqrt{\frac{16}{9} + 4} = \frac{2}{3}\sqrt{13} \text{ units.}$$

EXAMPLE 3 Reduce the equation $\sqrt{3}x + y + 2 = 0$ to the normal form

$x \cos \alpha + y \sin \alpha = p$, and hence find the values of α and p .

SOLUTION We have

$$\sqrt{3}x + y + 2 = 0 \Rightarrow -\sqrt{3}x - y = 2 \quad [\text{keeping constant +ve}]$$

$$\Rightarrow \left(\frac{-\sqrt{3}}{2}\right)x + \left(\frac{-1}{2}\right)y = 1$$

$$[\text{on dividing throughout by } \sqrt{(\sqrt{3})^2 + 1^2}]$$

$$\Rightarrow x \cos \alpha + y \sin \alpha = p,$$

$$\text{where } \cos \alpha = \frac{-\sqrt{3}}{2}, \sin \alpha = \frac{-1}{2} \text{ and } p = 1.$$

Since $\cos \alpha < 0$ and $\sin \alpha < 0$, α lies in the third quadrant.

$$\text{Now, } \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \left(\frac{-1}{2}\right) \times \left(\frac{-2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \tan (180^\circ + 30^\circ) = \tan 210^\circ$$

$$\Rightarrow \alpha = 210^\circ.$$

Thus, $\alpha = 210^\circ$ and $p = 1$.

Hence, the given equation in normal form is given by

$$x \cos 210^\circ + y \sin 210^\circ = 1.$$

EXAMPLE 4 Reduce the equation $x + \sqrt{3}y + 5 = 0$ to the normal form

$x \cos \alpha + y \sin \alpha = p$, and hence find the values of α and p .

SOLUTION We have

$$x + \sqrt{3}y + 5 = 0 \Rightarrow -x - \sqrt{3}y = 5 \quad [\text{keeping constant +ve}]$$

$$\Rightarrow \left(\frac{-1}{2}\right)x + \left(\frac{-\sqrt{3}}{2}\right)y = \frac{5}{2}$$

$$[\text{on dividing throughout by } \sqrt{(-1)^2 + (-\sqrt{3})^2}]$$

$$\Rightarrow x \cos \alpha + y \sin \alpha = p,$$

$$\text{where } \cos \alpha = \frac{-1}{2}, \sin \alpha = \frac{-\sqrt{3}}{2} \text{ and } p = \frac{5}{2}.$$



Since $\cos \alpha < 0$ and $\sin \alpha < 0$, α lies in third quadrant.

$$\text{Now, } \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \left(\frac{-\sqrt{3}}{2} \right) \times (-2) = \sqrt{3} = \tan (180^\circ + 60^\circ) = \tan 240^\circ$$

$$\therefore \alpha = 240^\circ.$$

$$\text{Thus, } \alpha = 240^\circ \text{ and } p = \frac{5}{2}.$$

Hence, the given equation in normal form is

$$x \cos 240^\circ + y \sin 240^\circ = \frac{5}{2}.$$

EXAMPLE 5 Reduce the equation $y + 4 = 0$ to the normal form $x \cos \alpha + y \sin \alpha = p$, and hence find the values of α and p .

SOLUTION We have

$$\begin{aligned} y + 4 = 0 &\Rightarrow -y = 4 && \text{[keeping constant positive]} \\ &\Rightarrow x \cos \alpha + y \sin \alpha = p, \end{aligned}$$

$$\text{where } \cos \alpha = 0, \sin \alpha = -1 \text{ and } p = 4.$$

$$\text{Now, } (\cos \alpha = 0 \text{ and } \sin \alpha = -1) \Rightarrow \alpha = 270^\circ.$$

Hence, the required normal form of the given equation is

$$x \cos 270^\circ + y \sin 270^\circ = 4.$$

$$\text{Clearly, } \alpha = 270^\circ \text{ and } p = 4.$$

EXERCISE 20G

1. Reduce the equation $2x - 3y - 5 = 0$ to slope-intercept form, and find from it the slope and y -intercept.
2. Reduce the equation $5x + 7y - 35 = 0$ to slope-intercept form, and hence find the slope and the y -intercept of the line.
3. Reduce the equation $y + 5 = 0$ to slope-intercept form, and hence find the slope and the y -intercept of the line.
4. Reduce the equation $3x - 4y + 12 = 0$ to intercepts form. Hence, find the length of the portion of the line intercepted between the axes.
5. Reduce the equation $5x - 12y = 60$ to intercepts form. Hence, find the length of the portion of the line intercepted between the axes.
6. Find the inclination of the line:
 - (i) $x + \sqrt{3}y + 6 = 0$
 - (ii) $3x + 3y + 8 = 0$
 - (iii) $\sqrt{3}x - y - 4 = 0$
7. Reduce the equation $x + y - \sqrt{2} = 0$ to the normal form $x \cos \alpha + y \sin \alpha = p$, and hence find the values of α and p .
8. Reduce the equation $x + \sqrt{3}y - 4 = 0$ to the normal form $x \cos \alpha + y \sin \alpha = p$, and hence find the values of α and p .
9. Reduce each of the following equations to normal form:
 - (i) $x + y - 2 = 0$
 - (ii) $x + y + \sqrt{2} = 0$
 - (iii) $x + 5 = 0$
 - (iv) $2y - 3 = 0$
 - (v) $4x + 3y - 9 = 0$



1. $y = \frac{2}{3}x - \frac{5}{3}, m = \frac{2}{3}$ and $c = -\frac{5}{3}$
2. $y = \frac{-5}{7}x + 5, m = \frac{-5}{7}$ and $c = 5$
3. $y = 0 \cdot x - 5, m = 0$ and $c = -5$
4. $\frac{x}{-4} + \frac{y}{3} = 1, 5$ units
5. $\frac{x}{12} + \frac{y}{-5} = 1, 13$ units
6. (i) 150° (ii) 135° (iii) 60°
7. $x \cos 45^\circ + y \sin 45^\circ = 1; \alpha = 45^\circ, p = 1$
8. $x \cos 60^\circ + y \sin 60^\circ = 2; \alpha = 60^\circ, p = 2$
9. (i) $x \cos 45^\circ + y \sin 45^\circ = \sqrt{2}$ (ii) $x \cos 225^\circ + y \sin 225^\circ = 1$
 (iii) $x \cos 180^\circ + y \sin 180^\circ = 5$ (iv) $x \cos 90^\circ + y \sin 90^\circ = \frac{3}{2}$
 (v) $x \cos \alpha + y \sin \alpha = p$, where $\cos \alpha = \frac{4}{5}, \sin \alpha = \frac{3}{5}$ and $p = \frac{9}{5}$

DISTANCE OF A POINT FROM A LINE

THEOREM 3 Prove that the length of perpendicular from a given point $P(x_1, y_1)$ on a line $Ax + By + C = 0$, is given by

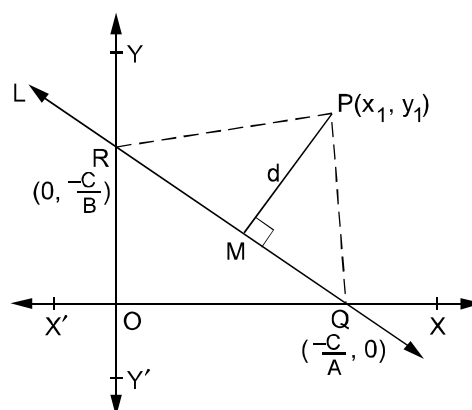
$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

PROOF Let L be the line represented by the equation $Ax + By + C = 0$ and let $P(x, y)$ be a given point outside it.

Let this line L intersect the x -axis and y -axis at the points Q and R respectively, and let $PM \perp QR$.

Now,

$$\begin{aligned} Ax + By + C = 0 &\Rightarrow Ax + By = -C \\ &\Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1 \\ &\Rightarrow \frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1. \end{aligned}$$



Thus, the given line intersects the x -axis and the y -axis at the points

$$Q\left(\frac{-C}{A}, 0\right) \text{ and } R\left(0, \frac{-C}{B}\right).$$

Draw $PM \perp QR$.

$$\begin{aligned}
 \text{Area of } \triangle PQR &= \frac{1}{2} \cdot \left| x_1 \left(\frac{-C}{B} - 0 \right) + 0 \cdot (0 - y_1) + \left(\frac{-C}{A} \right) \cdot \left(y_1 + \frac{C}{B} \right) \right| \\
 &= \frac{1}{2} \cdot \left| -C \left(\frac{x_1}{B} + \frac{y_1}{A} + \frac{C}{AB} \right) \right| = \frac{1}{2} \cdot \left| \frac{C}{AB} (Ax_1 + By_1 + C) \right| \\
 &= \left| \frac{C}{2AB} (Ax_1 + By_1 + C) \right| \quad \dots (i)
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, area of } \triangle PQR &= \frac{1}{2} \times QR \times PM = \frac{1}{2} \times PM \times \sqrt{(OQ)^2 + (OR)^2} \\
 &= \frac{1}{2} \times PM \times \sqrt{\left(\frac{-C}{A} \right)^2 + \left(\frac{-C}{B} \right)^2} \\
 &\quad \left[\because OQ = \frac{-C}{A} \text{ and } OR = \frac{-C}{B} \right] \\
 &= \frac{1}{2} \times PM \times \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} \quad \dots (ii)
 \end{aligned}$$

From (i) and (ii), we get

$$\begin{aligned}
 \frac{1}{2} \times PM \times \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} &= \left| \frac{C}{2AB} \cdot (Ax_1 + By_1 + C) \right| \\
 \Rightarrow \frac{1}{2} \times \frac{C}{AB} \times \sqrt{A^2 + B^2} \times PM &= \frac{C}{2AB} \cdot |Ax_1 + By_1 + C| \\
 \Rightarrow PM &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.
 \end{aligned}$$

$$\text{Hence, } d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}, \text{ where } PM = d.$$

REMARK Clearly, the length of perpendicular from the origin on the line

$$Ax + By + C = 0 \text{ is } \frac{|C|}{\sqrt{A^2 + B^2}}.$$

DISTANCE BETWEEN TWO PARALLEL LINES

THEOREM 4 Prove that the distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$$

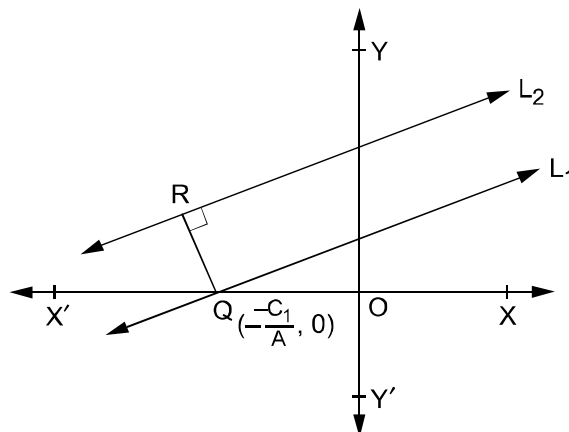
SOLUTION Let L_1 and L_2 be two given parallel lines represented by

$$Ax + By + C_1 = 0 \quad \dots (i)$$

$$\text{and } Ax + By + C_2 = 0. \quad \dots (ii)$$

$$\text{Putting } y = 0 \text{ in (i), we get } x = \left(-\frac{C_1}{A} \right).$$





Thus, L_1 intersects the x -axis at $Q = \left(-\frac{C_1}{A}, 0\right)$.

Let $QR \perp L_2$ and let $d = |QR|$. Then,

$d = \text{length of perpendicular from } Q\left(-\frac{C_1}{A}, 0\right) \text{ on } Ax + By + C_2 = 0$

$$\Rightarrow d = \frac{\left|A \times \left(-\frac{C_1}{A}\right) + B \times 0 + C_2\right|}{\sqrt{A^2 + B^2}} = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$$

REMARK Distance between two parallel lines $y = mx + C_1$ and $y = mx + C_2$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}.$$

SUMMARY

- (i) Length of perpendicular from a point $P(x_1, y_1)$ on the line $Ax + By + C = 0$ is given by, $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$
- (ii) Distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by, $d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$
- (iii) Distance between $y = mx + C_1$ and $y = mx + C_2$ is given by, $d = \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}.$

SOLVED EXAMPLES

EXAMPLE 1 Find the distance of the point $(4, 1)$ from the line $3x - 4y + 12 = 0$.

SOLUTION Clearly, the required distance is the length of perpendicular from the point $P(4, 1)$ on the line $3x - 4y + 12 = 0$.



Let the required distance be d . Then,

$$d = \frac{|3 \times 4 - 4 \times 1 + 12|}{\sqrt{3^2 + (-4)^2}} = \frac{20}{5} \text{ units} = 4 \text{ units.}$$

EXAMPLE 2 Find the distance of the point $(-1, 1)$ from the line $12(x + 6) = 5(y - 2)$.

SOLUTION The given point is $P(-1, 1)$ and the given line is $12x - 5y + 82 = 0$.

Let the required distance be d . Then,

$$\begin{aligned} d &= \text{length of perpendicular from } P(-1, 1) \text{ on the line } 12x - 5y + 82 = 0 \\ &= \frac{|12 \times (-1) - 5 \times 1 + 82|}{\sqrt{(12)^2 + (-5)^2}} = \frac{65}{\sqrt{169}} = \frac{65}{13} = 5 \text{ units.} \end{aligned}$$

EXAMPLE 3 Find the length of perpendicular from the point (a, b) to the line $\frac{x}{a} + \frac{y}{b} = 1$.

SOLUTION The given point is $P(a, b)$ and the given line is $bx + ay - ab = 0$.

Let d be the length of perpendicular from $P(a, b)$ to the line $bx + ay - ab = 0$.

$$\text{Then, } d = \frac{|b \times a + a \times b - ab|}{\sqrt{b^2 + a^2}} = \frac{|ab|}{\sqrt{a^2 + b^2}} \text{ units.}$$

EXAMPLE 4 Find the length of perpendicular from the origin to the line $4x + 3y - 2 = 0$.

SOLUTION The given point is $P(0, 0)$ and the given line is $4x + 3y - 2 = 0$.

Let d be the length of perpendicular from $P(0, 0)$ to the line $4x + 3y - 2 = 0$.

$$\text{Then, } d = \frac{|4 \times 0 + 3 \times 0 - 2|}{\sqrt{4^2 + 3^2}} = \frac{2}{5} \text{ unit.}$$

EXAMPLE 5 If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

SOLUTION The equation of the line making intercepts a and b on the axes is given by

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ i.e., } \frac{x}{a} + \frac{y}{b} - 1 = 0. \quad \dots (i)$$

Since p is the length of perpendicular from $O(0, 0)$ to line (i), we have

$$p = \frac{\left| \frac{1}{a} \times 0 + \frac{1}{b} \times 0 - 1 \right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$



$$\Rightarrow p^2 = \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \frac{a^2 b^2}{(b^2 + a^2)}$$

$$\Rightarrow \frac{1}{p^2} = \frac{(b^2 + a^2)}{a^2 b^2} = \left(\frac{b^2}{a^2 b^2} + \frac{a^2}{a^2 b^2}\right) = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

$$\text{Hence, } \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

EXAMPLE 6 Find the perpendicular distance of the line joining the points $A(\cos \theta, \sin \theta)$ and $B(\cos \phi, \sin \phi)$ from the origin.

SOLUTION We know that the equation of the line AB is given by

$$\frac{y - \sin \theta}{x - \cos \theta} = \frac{\sin \phi - \sin \theta}{\cos \phi - \cos \theta}$$

$$\Rightarrow \frac{y - \sin \theta}{x - \cos \theta} = \frac{2 \cos \left(\frac{\phi + \theta}{2}\right) \sin \left(\frac{\phi - \theta}{2}\right)}{-2 \sin \left(\frac{\phi + \theta}{2}\right) \sin \left(\frac{\phi - \theta}{2}\right)}$$

$$\Rightarrow \frac{y - \sin \theta}{x - \cos \theta} = \frac{\cos \left(\frac{\phi + \theta}{2}\right)}{-\sin \left(\frac{\phi + \theta}{2}\right)}$$

$$\Rightarrow x \cos \left(\frac{\phi + \theta}{2}\right) + y \sin \left(\frac{\phi + \theta}{2}\right) - \left\{ \cos \theta \cdot \cos \left(\frac{\phi + \theta}{2}\right) + \sin \theta \cdot \sin \left(\frac{\phi + \theta}{2}\right) \right\} = 0$$

$$\Rightarrow x \cos \left(\frac{\phi + \theta}{2}\right) + y \sin \left(\frac{\phi + \theta}{2}\right) - \cos \left(\frac{\phi + \theta}{2} - \theta\right) = 0$$

$$\Rightarrow x \cos \left(\frac{\phi + \theta}{2}\right) + y \sin \left(\frac{\phi + \theta}{2}\right) - \cos \left(\frac{\phi - \theta}{2}\right) = 0. \quad \dots (i)$$

Let d be the perpendicular distance from the origin to line (i). Then,

$$d = \frac{\left| 0 \times \cos \left(\frac{\phi + \theta}{2}\right) + 0 \times \sin \left(\frac{\phi + \theta}{2}\right) - \cos \left(\frac{\phi - \theta}{2}\right) \right|}{\sqrt{\cos^2 \left(\frac{\phi + \theta}{2}\right) + \sin^2 \left(\frac{\phi + \theta}{2}\right)}} = \left| \cos \left(\frac{\phi - \theta}{2}\right) \right|.$$

Hence, the required distance is $\left| \cos \left(\frac{\phi - \theta}{2}\right) \right|$.



EXAMPLE 7 If p_1 and p_2 are the lengths of perpendiculars from the origin to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ and $x \cos \theta - y \sin \theta = a \cos 2\theta$ respectively then prove that $4p_1^2 + p_2^2 = a^2$.

SOLUTION We have

$$x \sec \theta + y \operatorname{cosec} \theta = a \Rightarrow \frac{x}{\cos \theta} + \frac{y}{\sin \theta} - a = 0. \quad \dots (i)$$

Now, p_1 is the length of perpendicular from the origin to line (i), so we have

$$p_1 = \frac{\left| \frac{1}{\cos \theta} \times 0 + \frac{1}{\sin \theta} \times 0 - a \right|}{\sqrt{\frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}}} = |a| \sin \theta \cos \theta = \frac{|a|}{2} \sin 2\theta$$

$$\Rightarrow 2p_1 = |a| \sin 2\theta$$

$$\Rightarrow 4p_1^2 = a^2 \sin^2 2\theta. \quad \dots (ii)$$

$$\text{The other line is } x \cos \theta - y \sin \theta - a \cos 2\theta = 0. \quad \dots (iii)$$

Now, p_2 is the length of perpendicular from the origin to line (ii), so we have

$$p_2 = \frac{|0 \times \cos \theta - 0 \times \sin \theta - a \cos 2\theta|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = |a \cos 2\theta|$$

$$\Rightarrow p_2^2 = a^2 \cos^2 2\theta. \quad \dots (iv)$$

Adding (ii) and (iv), we get

$$4p_1^2 + p_2^2 = a^2(\sin^2 2\theta + \cos^2 2\theta)$$

$$\Rightarrow 4p_1^2 + p_2^2 = a^2.$$

EXAMPLE 8 What are the points on the y -axis whose perpendicular distance from the line $\frac{x}{3} - \frac{y}{4} = 1$ is 3 units?

SOLUTION $\frac{x}{3} - \frac{y}{4} = 1 \Rightarrow 4x - 3y - 12 = 0. \quad \dots (i)$

Let $P(0, y)$ be a point on the y -axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3 units. Then,

$$\frac{|4 \times 0 - 3y - 12|}{\sqrt{4^2 + (-3)^2}} = 3 \Rightarrow |-3y - 12| = 15 \Rightarrow |-3(y + 4)| = 15$$

$$\Rightarrow |-(y + 4)| = 5 \Rightarrow |y + 4| = 5$$

$$\Rightarrow (y + 4 = 5) \text{ or } (y + 4 = -5)$$

$$\Rightarrow y = 1 \text{ or } y = -9.$$

Hence, the required points are $(0, 1)$ and $(0, -9)$.

EXAMPLE 9 Find the distance between the parallel lines $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$.



SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$15x + 8y - 34 = 0 \Rightarrow y = \frac{-15}{8}x + \frac{17}{4} \quad \dots (i)$$

$$15x + 8y + 31 = 0 \Rightarrow y = \frac{-15}{8}x - \frac{31}{8} \quad \dots (ii)$$

Clearly, the slopes of the given lines are equal and so they are parallel.

The given lines are of the form $y = mx + C_1$ and $y = mx + C_2$, where $m = \frac{-15}{8}$, $C_1 = \frac{17}{4}$ and $C_2 = \frac{-31}{8}$.

$\therefore$ distance between the given lines

$$\begin{aligned} &= \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}, \text{ where } m = \frac{-15}{8}, C_1 = \frac{17}{4} \text{ and } C_2 = \frac{-31}{8} \\ &= \frac{\left| \frac{-31}{8} - \frac{17}{4} \right|}{\sqrt{1 + \left(\frac{-15}{8} \right)^2}} = \frac{\left| \frac{-65}{8} \right|}{\sqrt{1 + \frac{225}{64}}} = \frac{\left(\frac{65}{8} \right)}{\sqrt{\frac{289}{64}}} = \left(\frac{65}{8} \times \frac{8}{17} \right) = \frac{65}{17} \text{ units.} \end{aligned}$$

Hence, the distance between the given lines is $\frac{65}{17}$ units.

EXAMPLE 10 Find the distance between the lines $3x - 4y + 9 = 0$ and $6x - 8y - 17 = 0$.

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$3x - 4y + 9 = 0 \Rightarrow y = \frac{3}{4}x + \frac{9}{4} \quad \dots (i)$$

$$6x - 8y - 17 = 0 \Rightarrow y = \frac{3}{4}x - \frac{17}{8} \quad \dots (ii)$$

Clearly, the slopes of the given lines are equal and so they are parallel.

Let the given lines be $y = mx + C_1$ and $y = mx + C_2$, where $m = \frac{3}{4}$, $C_1 = \frac{9}{4}$ and $C_2 = \frac{-17}{8}$.

$$\begin{aligned} \therefore \text{ distance between given lines} &= \frac{|C_2 - C_1|}{\sqrt{1 + m^2}} \\ &= \frac{\left| \frac{-17}{8} - \frac{9}{4} \right|}{\sqrt{1 + \frac{9}{16}}} = \frac{\left| \frac{-35}{8} \right|}{\left(\frac{5}{4} \right)} = \left(\frac{35}{8} \times \frac{4}{5} \right) \\ &= \frac{7}{2} \text{ units.} \end{aligned}$$

Hence, the distance between the given lines is $\frac{7}{2}$ units.



EXAMPLE 11 Prove that the line $5x - 2y - 1 = 0$ is mid-parallel to the lines $5x - 2y - 9 = 0$ and $5x - 2y + 7 = 0$.

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$5x - 2y - 1 = 0 \Rightarrow y = \frac{5}{2}x - \frac{1}{2} \quad \dots (i)$$

$$5x - 2y - 9 = 0 \Rightarrow y = \frac{5}{2}x - \frac{9}{2} \quad \dots (ii)$$

$$5x - 2y + 7 = 0 \Rightarrow y = \frac{5}{2}x + \frac{7}{2} \quad \dots (iii)$$

Clearly, the slope of (i) is equal to the slope of each of (ii) and (iii).

So, the line (i) is parallel to each of the given lines (ii) and (iii).

Let the given lines be $y = mx + C$, $y = mx + C_1$ and $y = mx + C_2$ respectively. Then, $m = \frac{5}{2}$, $C = -\frac{1}{2}$, $C_1 = -\frac{9}{2}$ and $C_2 = \frac{7}{2}$.

Let d_1 and d_2 be the distances of (i) from (ii) and (iii) respectively.

$$\text{Then, } d_1 = \frac{|C_1 - C|}{\sqrt{1 + m^2}} = \frac{\left| -\frac{9}{2} + \frac{1}{2} \right|}{\sqrt{1 + \frac{25}{4}}} = |-4| \cdot \frac{2}{\sqrt{29}} = \frac{(4 \times 2)}{\sqrt{29}} = \frac{8}{\sqrt{29}} \text{ units}$$

$$\text{and } d_2 = \frac{|C_2 - C|}{\sqrt{1 + m^2}} = \frac{\left| \frac{7}{2} + \frac{1}{2} \right|}{\sqrt{1 + \frac{25}{4}}} = \left(4 \times \frac{2}{\sqrt{29}} \right) = \frac{8}{\sqrt{29}} \text{ units.}$$

Thus, $d_1 = d_2$.

This shows that (i) is equidistant from (ii) and (iii).

Hence, $5x - 2y - 1 = 0$ is mid-parallel to the lines $5x - 2y - 9 = 0$ and $5x - 2y + 7 = 0$.

EXAMPLE 12 Find the equation of the line midway between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$9x + 6y - 7 = 0 \Rightarrow y = \frac{-3}{2}x + \frac{7}{6} \quad \dots (i)$$

$$3x + 2y + 6 = 0 \Rightarrow y = \frac{-3}{2}x - 3 \quad \dots (ii)$$

Clearly, the slope of each one of the given lines is $\frac{-3}{2}$.

Let the given lines be $y = mx + C_1$ and $y = mx + C_2$.

Then, $m = \frac{-3}{2}$, $C_1 = \frac{7}{6}$ and $C_2 = -3$.



Let L be the required line. Then, L is parallel to each one of (i) and (ii), and equidistant from each one of them.

$$\therefore \text{ slope of } L = \frac{-3}{2}.$$

$$\text{Let the equation of } L \text{ be } y = \frac{-3}{2}x + C \quad \dots \text{ (iii)}$$

Then, distance between (i) and (iii) must be equal to the distance between (ii) and (iii).

$$\begin{aligned} \therefore \frac{|C_1 - C|}{\sqrt{1 + m^2}} &= \frac{|C_2 - C|}{\sqrt{1 + m^2}} \Rightarrow |C_1 - C| = |C_2 - C| \\ &\Rightarrow \left| \frac{7}{6} - C \right| = |-3 - C| \Rightarrow \left| \frac{7}{6} - C \right| = |3 + C| \\ &\Rightarrow \frac{7}{6} - C = 3 + C \Rightarrow 2C = \frac{-11}{6} \Rightarrow C = \frac{-11}{12}. \end{aligned}$$

$$\therefore \text{ equation of } L \text{ is } y = \frac{-3}{2}x - \frac{11}{12}, \text{ i.e., } 18x + 12y + 11 = 0.$$

Hence, the line $18x + 12y + 11 = 0$ is midway between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

EXAMPLE 13 Find the coordinates of a point on the line $x + y + 3 = 0$, whose distance from the line $x + 2y + 2 = 0$ is $\sqrt{5}$.

SOLUTION Let the required point be $P(a, b)$. Then,

$$a + b + 3 = 0. \quad \dots \text{ (i)}$$

$$\begin{aligned} \text{Also, } \frac{|a + 2b + 2|}{\sqrt{1^2 + 2^2}} &= \sqrt{5} \Rightarrow |a + 2b + 2| = 5 \\ &\Rightarrow (a + 2b + 2 = 5) \quad \text{or} \quad (a + 2b + 2 = -5) \\ &\Rightarrow a + 2b = 3 \quad \text{or} \quad a + 2b = -7. \end{aligned}$$

$$\text{Thus, } a + 2b = 3 \quad \dots \text{ (ii)} \quad \text{and} \quad a + 2b = -7 \quad \dots \text{ (iii)}$$

On solving (i) and (ii), we get $a = -9$ and $b = 6$.

On solving (i) and (iii), we get $a = 1$ and $b = -4$.

Hence, the required points are $A(-9, 6)$ and $B(1, -4)$.

EXERCISE 20H

- Find the distance of the point $(3, -5)$ from the line $3x - 4y = 27$.
- Find the distance of the point $(-2, 3)$ from the line $12x = 5y + 13$.
- Find the distance of the point $(-4, 3)$ from the line $4(x + 5) = 3(y - 6)$.
- Find the distance of the point $(2, 3)$ from the line $y = 4$.
- Find the distance of the point $(4, 2)$ from the line joining the points $(4, 1)$ and $(2, 3)$.



6. Find the length of perpendicular from the origin to each of the following lines:
 (i) $7x + 24y = 50$ (ii) $4x + 3y = 9$ (iii) $x = 4$
7. Prove that the product of the lengths of perpendiculars drawn from the points $A(\sqrt{a^2 - b^2}, 0)$ and $B(-\sqrt{a^2 - b^2}, 0)$ to the line $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$, is b^2 .
8. Find the values of k for which the length of perpendicular from the point $(4, 1)$ on the line $3x - 4y + k = 0$ is 2 units.
9. Show that the length of perpendicular from the point $(7, 0)$ to the line $5x + 12y - 9 = 0$ is double the length of perpendicular to it from the point $(2, 1)$.
10. The points $A(2, 3)$, $B(4, -1)$ and $C(-1, 2)$ are the vertices of $\triangle ABC$. Find the length of perpendicular from C on AB and hence find the area of $\triangle ABC$.
11. What are the points on the x -axis whose perpendicular distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units?
12. Find all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y = 10$.
13. A vertex of a square is at the origin and its one side lies along the line $3x - 4y - 10 = 0$. Find the area of the square.
14. Find the distance between the parallel lines $4x - 3y + 5 = 0$ and $4x - 3y + 7 = 0$.
15. Find the distance between the parallel lines $8x + 15y - 36 = 0$ and $8x + 15y + 32 = 0$.
16. Find the distance between the parallel lines $y = mx + c$ and $y = mx + d$.
17. Find the distance between the parallel lines $p(x + y) + q = 0$ and $p(x + y) - r = 0$.
18. Prove that the line $12x - 5y - 3 = 0$ is mid-parallel to the lines $12x - 5y + 7 = 0$ and $12x - 5y - 13 = 0$.
19. The perpendicular distance of a line from the origin is 5 units and its slope is -1 . Find the equation of the line.

ANSWERS (EXERCISE 20H)

1. $\frac{2}{5}$ unit
2. 4 units
3. $\frac{13}{5}$ units
4. 1 unit
5. $\frac{\sqrt{2}}{2}$ units
6. (i) 2 units (ii) $\frac{9}{5}$ units (iii) 4 units
8. $k = 2$ or $k = -18$
10. $\frac{7}{\sqrt{5}}$ units, 7 sq units
11. $(8, 0)$ and $(-2, 0)$



12. (3, 1) and (-7, 11)

13. 4 sq units

14. $\frac{2}{5}$ units

15. 4 units

16. $\frac{|d-c|}{\sqrt{1+m^2}}$

17. $\frac{|q+r|}{\sqrt{2p}}$

19. $x + y + 5\sqrt{2} = 0$ or $x + y - 5\sqrt{2} = 0$

HINTS TO SOME SELECTED QUESTIONS1. The given line is $0 \cdot x + 1 \cdot y - 4 = 0$.

5. Equation of the line joining the points A(4, 1) and B(2, 3) is

$$\frac{y-1}{x-4} = \frac{3-1}{2-4}, \quad \text{i.e., } x + y - 5 = 0.$$

6. (iii) The given line is $x + 0 \cdot y - 4 = 0$.7. Put $\sqrt{a^2 - b^2} = c$. Then,

$$\begin{aligned} (d_1 \times d_2) &= \frac{\left| \frac{c \cos \theta}{a} + 0 - 1 \right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \times \frac{\left| \frac{-c \cos \theta}{a} + 0 - 1 \right|}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} = \frac{\left| \frac{c \cos \theta}{a} - 1 \right| \left| \frac{c \cos \theta}{a} + 1 \right|}{\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)} \\ &= \frac{\left| \frac{c^2 \cos^2 \theta}{a^2} - 1 \right|}{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)} \times a^2 b^2 = \frac{|(a^2 - b^2) \cos^2 \theta - a^2|}{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)} \times b^2 \\ &= \frac{|a^2(1 - \sin^2 \theta) - b^2 \cos^2 \theta - a^2|}{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)} \times b^2 = \frac{|-(a^2 \sin^2 \theta + b^2 \cos^2 \theta)|}{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)} \times b^2 \\ &= b^2. \end{aligned}$$

$$8. \frac{|3 \times 4 - 4 \times 1 + k|}{\sqrt{3^2 + (-4)^2}} = 2 \Leftrightarrow |8 + k| = 10 \Leftrightarrow (8 + k) = 10 \quad \text{or} \quad (8 + k) = -10.$$

$$10. \text{Equation of } AB \text{ is } \frac{y-3}{x-2} = \frac{-4}{2} \Rightarrow 2x + y - 7 = 0.$$

Length of perpendicular from C(-1, 2) to $2x + y - 7 = 0$ is equal to $\frac{|2 \times (-1) + 2 - 7|}{\sqrt{2^2 + 1^2}} = \frac{7}{\sqrt{5}}$ units.

$$\text{ar}(\triangle ABC) = \frac{1}{2} \times AB \times \frac{7}{\sqrt{5}} = \left(\frac{1}{2} \times \sqrt{20} \times \frac{7}{\sqrt{5}} \right) \text{sq units} = 7 \text{ sq units}.$$

11. Let the required point be P(x, 0).

Given line is $4x + 3y - 12 = 0$.

$$\text{Now } \frac{|4x + 3 \times 0 - 12|}{\sqrt{4^2 + 3^2}} = 4 \Leftrightarrow |x - 3| = 5 \Rightarrow x - 3 = 5 \quad \text{or} \quad x - 3 = -5$$

$$\Rightarrow x = 8 \quad \text{or} \quad x = -2.$$



12. Let $x = a$. Then, $x + y = 4 \Rightarrow y = (4 - a)$.

Let the required point be $P(a, 4 - a)$. Then,

$$\frac{|4a + 3(4 - a) - 10|}{\sqrt{4^2 + 3^2}} = 1 \Rightarrow |a + 2| = 5 \Rightarrow a + 2 = 5 \quad \text{or} \quad a + 2 = -5$$

$$\Rightarrow a = 3 \quad \text{or} \quad a = -7.$$

$\therefore$ required points are $(3, 1)$ and $(-7, 11)$.

13. Side of the square = length of perp. from $(0, 0)$ on $3x - 4y - 10 = 0$

$$= \frac{|3 \times 0 - 4 \times 0 - 10|}{\sqrt{3^2 + (-4)^2}} = 2 \text{ units.}$$

Area of the square = (2×2) sq units = 4 sq units.

16. Putting $x = 0$ in $y = mx + c$, we get $y = c$.

Thus, $P(0, c)$ is a point on $y = mx + c$.

Required distance = length of perp. from $P(0, c)$ on $y = mx + d$

$$= \frac{|m \times 0 - c + d|}{\sqrt{1 + m^2}} = \frac{|d - c|}{\sqrt{1 + m^2}}.$$

17. Putting $x = 0$ in $px + py + q = 0$, we get $y = (-q/p)$.

Required distance = length of perp. from $P\left(0, \frac{-q}{p}\right)$ on $px + py - r = 0$

$$= \frac{\left|p \times 0 + p \times \left(\frac{-q}{p}\right) - r\right|}{\sqrt{p^2 + p^2}} = \frac{|-q - r|}{\sqrt{2}p} = \frac{|q + r|}{\sqrt{2}p}.$$

19. Let the required equation be $y = -x + c$.

$$\therefore \frac{|0 + 0 - c|}{\sqrt{1^2 + 1^2}} = 5 \Rightarrow |c| = 5\sqrt{2} \Rightarrow c = 5\sqrt{2} \quad \text{or} \quad c = -5\sqrt{2}.$$

MISCELLANEOUS PROBLEMS

It may be mentioned here that when the equations of two lines are given, then on solving them simultaneously, we get the point of intersection of the given lines.

EXAMPLE 1 Find the point of intersection of the lines $5x + 7y = 3$ and $2x - 3y = 7$.

SOLUTION The given equations are:

$$5x + 7y - 3 = 0 \quad \dots (i)$$

$$2x - 3y - 7 = 0 \quad \dots (ii)$$

On solving (i) and (ii), by cross multiplication, we get

$$\frac{x}{(-49 - 9)} = \frac{y}{(-6 + 35)} = \frac{1}{(-15 - 14)}$$



$$\Rightarrow x = \left(\frac{-58}{-29}\right) = 2 \quad \text{and} \quad y = \left(\frac{29}{-29}\right) = -1.$$

Hence, the point of intersection of the given lines is $P(2, -1)$.

EXAMPLE 2 Find the equation of the line parallel to the y -axis and drawn through the point of intersection of the lines $x - 7y + 15 = 0$ and $2x + y = 0$.

SOLUTION The given equations are:

$$x - 7y + 15 = 0 \quad \dots (i)$$

$$2x + y = 0 \quad \dots (ii)$$

On solving (i) and (ii), we get $x = -1$ and $y = 2$.

Thus, the given lines intersect at the point $P(-1, 2)$.

The line parallel to the y -axis and drawn through $P(-1, 2)$ is given by $x = -1$, i.e., $x + 1 = 0$.

Hence, the required equation is $x + 1 = 0$.

EXAMPLE 3 Find the equation of the line passing through the intersection of the lines $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$, and parallel to the line $y - x = 8$.

SOLUTION The given intersecting lines are:

$$x + 2y + 3 = 0 \quad \dots (i)$$

$$3x + 4y + 7 = 0 \quad \dots (ii)$$

By cross multiplication, we get

$$\frac{x}{(14 - 12)} = \frac{y}{(9 - 7)} = \frac{1}{(4 - 6)} \Rightarrow \frac{x}{2} = \frac{y}{2} = \frac{1}{-2}$$

$$\Rightarrow x = \frac{2}{-2} = -1 \quad \text{and} \quad y = \frac{2}{-2} = -1.$$

Thus, the lines (i) and (ii) intersect at the point $P(-1, -1)$.

$$\text{Now, } y - x = 8 \Rightarrow y = x + 8. \quad \dots (iii)$$

Slope of line (iii) is 1.

Slope of the line parallel to this line is 1.

$$\text{So, the equation of the required line is } \frac{y + 1}{x + 1} = 1 \Rightarrow x - y = 0.$$

EXAMPLE 4 Find the value of k for which the lines $3x + y = 2$, $kx + 2y = 3$ and $2x - y = 3$ may intersect at a point.

SOLUTION The given lines are:

$$3x + y - 2 = 0 \quad \dots (i)$$

$$kx + 2y - 3 = 0 \quad \dots (ii)$$

$$2x - y - 3 = 0 \quad \dots (iii)$$

On solving (i) and (iii) by cross multiplication, we get

$$\frac{x}{(-3 - 2)} = \frac{y}{(-4 + 9)} = \frac{1}{(-3 - 2)} \Rightarrow \frac{x}{-5} = \frac{y}{5} = \frac{1}{-5}$$

$$\Rightarrow x = \left(\frac{-5}{-5}\right) = 1 \quad \text{and} \quad y = \left(\frac{5}{-5}\right) = -1.$$

Thus, the point of intersection of (i) and (iii) is $P(1, -1)$.



For the given lines to intersect at a point, $x = 1$ and $y = -1$ must satisfy (ii) also.

$$\therefore (k \times 1) + 2 \times (-1) - 3 = 0 \Rightarrow k - 5 = 0 \Rightarrow k = 5.$$

Hence, $k = 5$.

EXAMPLE 5 Show that the lines $x - y = 6$, $4x - 3y = 20$ and $6x + 5y + 8 = 0$ are concurrent. Also find their point of intersection.

SOLUTION The given equations are:

$$x - y - 6 = 0 \quad \dots (i)$$

$$4x - 3y - 20 = 0 \quad \dots (ii)$$

$$6x + 5y + 8 = 0 \quad \dots (iii)$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{x}{(20 - 18)} = \frac{y}{(-24 + 20)} = \frac{1}{(-3 + 4)} \Rightarrow \frac{x}{2} = \frac{y}{-4} = \frac{1}{1} \Rightarrow x = 2, y = -4.$$

Thus, the lines (i) and (ii) intersect at the point $P(2, -4)$.

Putting $x = 2$ and $y = -4$ in (iii), we get

$$\text{LHS} = 6 \times 2 + 5 \times (-4) + 8 = 0 = \text{RHS}.$$

This shows that the point $P(2, -4)$ also lies on (iii).

Thus, all the given three lines intersect at the same point.

Hence, the given lines are concurrent and their point of intersection is $P(2, -4)$.

EXAMPLE 6 If the three lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent then show that

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0.$$

SOLUTION The given lines are:

$$m_1x - y + c_1 = 0 \quad \dots (i)$$

$$m_2x - y + c_2 = 0 \quad \dots (ii)$$

$$m_3x - y + c_3 = 0 \quad \dots (iii)$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{x}{(-c_2 + c_1)} = \frac{y}{(m_2c_1 - m_1c_2)} = \frac{1}{(-m_1 + m_2)} \\ \Rightarrow x = \frac{(c_1 - c_2)}{(m_2 - m_1)} \quad \text{and} \quad y = \frac{(m_2c_1 - m_1c_2)}{(m_2 - m_1)}.$$

Thus, the point of intersection of (i) and (ii) is $P \left(\frac{c_1 - c_2}{m_2 - m_1}, \frac{m_2c_1 - m_1c_2}{m_2 - m_1} \right)$.

Since the given three lines meet at a point, the point P must therefore lie on (iii) also.

$$\therefore m_3 \cdot \frac{(c_1 - c_2)}{(m_2 - m_1)} - \frac{(m_2c_1 - m_1c_2)}{(m_2 - m_1)} + c_3 = 0$$

$$\Rightarrow m_3(c_1 - c_2) - (m_2c_1 - m_1c_2) + c_3(m_2 - m_1) = 0$$

$$\Rightarrow m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0.$$



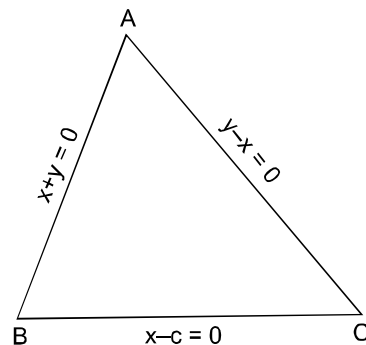
EXAMPLE 7 Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - c = 0$.

SOLUTION Let the sides AC , AB and BC of $\triangle ABC$ be represented by $y - x = 0$, $x + y = 0$ and $x - c = 0$ respectively.

On solving these equations pairwise, we get the points $A(0, 0)$, $B(c, -c)$ and $C(c, c)$.

$\therefore$ area of $\triangle ABC$ is given by

$$\begin{aligned}\Delta &= \frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= \frac{1}{2} \cdot |0 \cdot (-c - c) + c(c - 0) + c(0 + c)| \text{ sq units} \\ &= \left(\frac{1}{2} \times 2c^2 \right) \text{ sq units} = c^2 \text{ sq units.}\end{aligned}$$



Hence, the required area is c^2 sq units.

EXAMPLE 8 Find the area of the triangle formed by the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$.

SOLUTION Let the sides AC , AB and BC of $\triangle ABC$ be represented by the equations:

$$\begin{aligned}m_1x - y + c_1 &= 0 & \dots \text{ (i)} \\ m_2x - y + c_2 &= 0 & \dots \text{ (ii)} \\ x &= 0 & \dots \text{ (iii)}\end{aligned}$$

On solving (i) and (ii) by cross multiplication, we have

$$\begin{aligned}\frac{x}{(-c_2 + c_1)} &= \frac{y}{(m_2c_1 - m_1c_2)} = \frac{1}{(-m_1 + m_2)} \\ \Rightarrow x &= \frac{(c_1 - c_2)}{(m_2 - m_1)} \text{ and } y = \frac{(m_2c_1 - m_1c_2)}{(m_2 - m_1)}.\end{aligned}$$

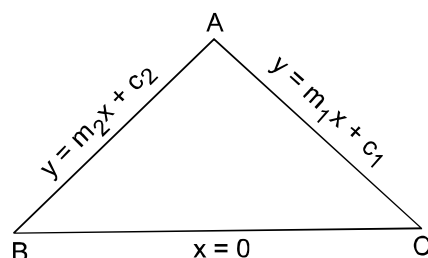
Thus, the lines AC and AB intersect at $A \left(\frac{c_1 - c_2}{m_2 - m_1}, \frac{m_2c_1 - m_1c_2}{m_2 - m_1} \right)$.

On solving (ii) and (iii), we get $B(0, c_2)$.

On solving (i) and (iii), we get $C(0, c_1)$.

$\therefore$ area of $\triangle ABC$ is given by

$$\begin{aligned}\Delta &= \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= \frac{1}{2} \cdot \left| \frac{(c_1 - c_2)}{(m_2 - m_1)} \cdot (c_2 - c_1) + 0 + 0 \right| = \frac{1}{2} \cdot \frac{(c_1 - c_2)^2}{|m_1 - m_2|}.\end{aligned}$$



Hence, the area of the triangle formed by the given lines is $\frac{1}{2} \cdot \frac{(c_1 - c_2)^2}{|m_1 - m_2|}$.

EXAMPLE 9 Find the image of the point $P(3, 8)$ with respect to the line $x + 3y = 7$, assuming the given line to be a plane mirror.

SOLUTION Let AB be the given line whose equation is $x + 3y - 7 = 0$ (i)

$$\text{Now, } x + 3y = 7 \Rightarrow y = -\frac{1}{3}x + \frac{7}{3}.$$

$$\text{Slope of } AB = -\frac{1}{3}.$$

Let $P(3, 8)$ be the given point and let $Q(x_1, y_1)$ be its image in AB . Join PQ , intersecting AB at N .

Then, $PN \perp AB$ and $PN = QN$.

Let the slope of PQ be m . Then,

$$m \times \left(-\frac{1}{3}\right) = -1 \Rightarrow m = 3 \quad [\because AB \perp PQ].$$

Equation of line PQ is given by

$$\frac{y - 8}{x - 3} = 3 \Rightarrow y - 8 = 3x - 9 \Rightarrow 3x - y - 1 = 0. \quad \dots (ii)$$

On solving (i) and (ii), we get

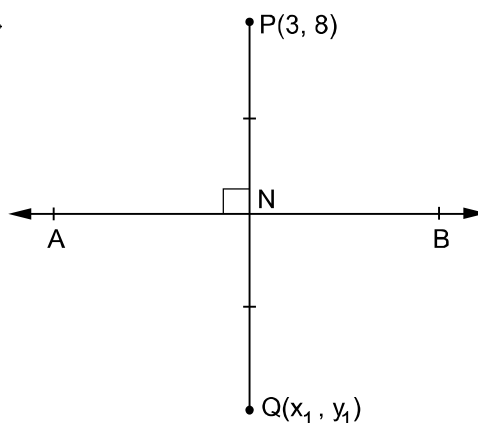
$$\frac{x}{(-3 - 7)} = \frac{y}{(-21 + 1)} = \frac{1}{(-1 - 9)} \Rightarrow \frac{x}{-10} = \frac{y}{-20} = \frac{1}{-10} \\ \Rightarrow x = 1 \quad \text{and} \quad y = 2.$$

Thus, AB and PQ intersect at the point $N(1, 2)$.

Also, N bisects PQ .

$$\therefore \left(\frac{3 + x_1}{2} = 1, \frac{8 + y_1}{2} = 2 \right) \Rightarrow x_1 = -1 \quad \text{and} \quad y_1 = -4.$$

Hence, the image of the given point $P(3, 8)$ is $Q(-1, -4)$.



EXERCISE 20I

1. Find the points of intersection of the lines

$$4x + 3y = 5 \quad \text{and} \quad x = 2y - 7.$$

2. Show that the lines $x + 7y = 23$ and $5x + 2y = 16$ intersect at the point $(2, 3)$.

3. Show that the lines $3x - 4y + 5 = 0$, $7x - 8y + 5 = 0$ and $4x + 5y = 45$ are concurrent. Also find their point of intersection.

4. Find the value of k so that the lines $3x - y - 2 = 0$, $5x + ky - 3 = 0$ and $2x + y - 3 = 0$ are concurrent.



5. Find the image of the point $P(1, 2)$ in the line $x - 3y + 4 = 0$.
6. Find the area of the triangle formed by the lines $x + y = 6$, $x - 3y = 2$ and $5x - 3y + 2 = 0$.
7. Find the area of the triangle formed by the lines $x = 0$, $y = 1$ and $2x + y = 2$.
8. Find the area of the triangle, the equations of whose sides are $y = x$, $y = 2x$ and $y - 3x = 4$.
9. Find the equation of the perpendicular drawn from the origin to the line $4x - 3y + 5 = 0$. Also, find the coordinates of the foot of the perpendicular.
10. Find the equation of the perpendicular drawn from the point $P(-2, 3)$ to the line $x - 4y + 7 = 0$. Also, find the coordinates of the foot of the perpendicular.
11. Find the equations of the medians of a triangle whose sides are given by the equations $3x + 2y + 6 = 0$, $2x - 5y + 4 = 0$ and $x - 3y - 6 = 0$.

ANSWERS (EXERCISE 20I)

1. $(-1, 3)$
2. $(5, 5)$
3. $k = -2$
4. $\left(\frac{6}{5}, \frac{7}{5}\right)$
5. 12 sq units
6. 1 sq unit
7. 4 sq units
8. $3x + 4y = 0, \left(-\frac{4}{5}, \frac{3}{5}\right)$
9. $4x + y + 5 = 0, \left(\frac{-27}{17}, \frac{23}{17}\right)$
10. $41x - 112y - 70 = 0, 16x - 59y - 120 = 0$ and $25x - 53y + 50 = 0$

SHIFTING OF ORIGIN

TRANSLATION OF AXES

Sometimes it is desirable to shift the origin to a new point and the new axes are transformed parallel to the original axes.

A transformation of this kind is known as translation of axes.

Let $P(x, y)$ be a point with reference to O as origin and OX and OY as the axes. Draw $PM \perp OX$, so that $OM = x$ and $PM = y$.

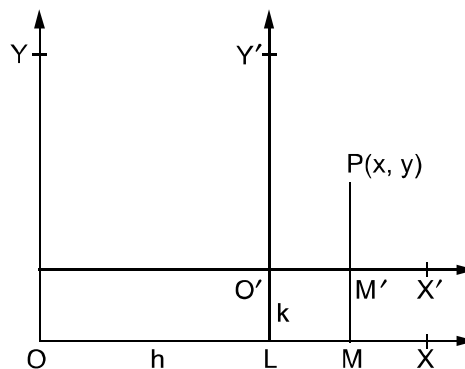
Let the new origin be $O'(h, k)$ referred to the old axes, i.e., $OL = h$ and $O'L = k$.

Draw $O'X' \parallel OX$, meeting PM at M' .

Also, draw $O'Y' \parallel OY$, meeting OM at L .

Now, we have the new origin at O' and the new axes as $O'X'$ and $O'Y'$.

Let $O'M' = x'$ and $M'P = y'$ with reference to the new axes. Then, $P(x', y')$ are the new coordinates of P .



Now, $OM = OL + LM \Rightarrow x = h + x' \Rightarrow x = x' + h \Rightarrow x' = x - h$.

And, $MP = MM' + PM' \Rightarrow y = k + y' \Rightarrow y = y' + k \Rightarrow y' = y - k$.

Hence, $x' = x - h$ and $y' = y - k$ are the relations between the new and old coordinates.

SOLVED EXAMPLES

EXAMPLE 1 If the origin be shifted to the point $(2, -3)$ by a translation of coordinate axes, find the new coordinates of the point $(4, 7)$.

SOLUTION Let the given point be $P(x, y)$. Then, $x = 4$ and $y = 7$.

Let the origin O be shifted to the point $O'(h, k)$, where $h = 2$, $k = -3$.

Then, the new coordinates of P are (x', y') , where

$$x' = x - h = (4 - 2) = 2$$

$$\text{and } y' = y - k = (7 + 3) = 10.$$

Hence, the coordinates of $(4, 7)$ in the new system are $(2, 10)$.

EXAMPLE 2 If the origin is shifted to the point $(2, 3)$, the coordinates of a point become $(5, -4)$. Find the original coordinates, when the axes are parallel.

SOLUTION Let the original coordinates be $P(x, y)$.

Let the origin O be shifted to the new point $O'(h, k)$, where $h = 2$ and $k = 3$.

Then, the new coordinates of P are (x', y') , where $x' = 5$ and $y' = -4$.

$$\text{Now, } x' = x - h \Rightarrow 5 = x - 2 \Rightarrow x = 7.$$

$$\text{And, } y' = y - k \Rightarrow -4 = y - 3 \Rightarrow y = -1.$$

Hence, the original coordinates are $(7, -1)$.

EXAMPLE 3 At what point the origin be shifted, if the coordinates of a point $(4, 5)$ become $(3, 7)$?

SOLUTION Let $O'(h, k)$ be the point to which the origin O be shifted.

Then, $(x = 4, y = 5)$ and $(x' = 3, y' = 7)$.

$$\text{Now, } x' = x - h \Rightarrow h = x - x' = (4 - 3) = 1.$$

$$\text{And, } y' = y - k \Rightarrow k = y - y' = (5 - 7) = -2.$$

Hence, the origin O must be shifted to the point $O'(1, -2)$.

EXAMPLE 4 If the origin be shifted to the point $(3, -1)$, find the new equation of the line $2x - 3y + 5 = 0$.

SOLUTION Let the origin O be shifted to the point $O'(h, k)$, where $h = 3$ and $k = -1$.

Let the new coordinates of $P(x, y)$ be $P(x', y')$.

$$\text{Then, } x' = x - h \Rightarrow x' = x - 3 \Rightarrow x = x' + 3.$$

$$\text{And, } y' = y - k \Rightarrow y' = y + 1 \Rightarrow y = y' - 1.$$

So, the new equation becomes:

$$2(x' + 3) - 3(y' - 1) + 5 = 0 \Rightarrow 2x' - 3y' + 14 = 0.$$



Hence, the equation of the straight line in the new system is $2x - 3y + 14 = 0$.

EXAMPLE 5 Find the point to which the origin be shifted after a translation, so that the equation $x^2 + y^2 - 4x - 8y + 3 = 0$ will have no first degree terms.

SOLUTION Let the origin O be shifted to a point $O'(h, k)$.

Let the new coordinates of $P(x, y)$ be $P(x', y')$.

Then, $x' = x - h \Rightarrow x = x' + h$.

And, $y' = y - k \Rightarrow y = y' + k$.

So, the new equation becomes:

$$\begin{aligned} & (x' + h)^2 + (y' + k)^2 - 4(x' + h) - 8(y' + k) + 3 = 0 \\ \Rightarrow & (x'^2 + h^2 + 2x'h) + (y'^2 + k^2 + 2y'k) - 4(x' + h) - 8(y' + k) + 3 = 0 \\ \Rightarrow & x'^2 + y'^2 + (2h - 4)x' + (2k - 8)y' + (h^2 + k^2 - 4h - 8k + 3) = 0. \end{aligned}$$

Since we are required to get an equation free from first degree terms, so we have:

$$\begin{aligned} & (2h - 4 = 0 \text{ and } 2k - 8 = 0) \\ \Rightarrow & (2h = 4 \text{ and } 2k = 8) \Rightarrow (h = 2 \text{ and } k = 4). \end{aligned}$$

Hence, the origin O should be shifted to the point $O'(2, 4)$.

EXERCISE 20J

1. If the origin is shifted to the point $(1, 2)$ by a translation of the axes, find the new coordinates of the point $(3, -4)$.
2. If the origin is shifted to the point $(-3, -2)$ by a translation of the axes, find the new coordinates of the point $(3, -5)$.
3. If the origin is shifted to the point $(0, -2)$ by a translation of the axes, the coordinates of a point become $(3, 2)$. Find the original coordinates of the point.
4. If the origin is shifted to the point $(2, -1)$ by a translation of the axes, the coordinates of a point become $(-3, 5)$. Find the original coordinates of the point.
5. At what point must the origin be shifted, if the coordinates of a point $(-4, 2)$ become $(3, -2)$?

Find what the given equation becomes when the origin is shifted to the point $(1, 1)$.

6. $x^2 + xy - 3x - y + 2 = 0$
7. $xy - y^2 - x + y = 0$
8. $x^2 - y^2 - 2x + 2y = 0$
9. $xy - x - y + 1 = 0$
10. Transform the equation $2x^2 + y^2 - 4x + 4y = 0$ to parallel axes when the origin is shifted to the point $(1, -2)$.

ANSWERS (EXERCISE 20J)

- | | | | |
|--------------|-------------------|-------------------|--------------------|
| 1. $(2, -6)$ | 2. $(6, -3)$ | 3. $(3, 0)$ | 4. $(-1, 4)$ |
| 5. $(-7, 4)$ | 6. $x^2 + xy = 0$ | 7. $y^2 - xy = 0$ | 8. $x^2 - y^2 = 0$ |



9. $xy = 0$

10. $2x^2 + y^2 - 6 = 0$

EQUATION OF FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES

Let the two intersecting lines L_1 and L_2 be given by

$$A_1x + B_1y + C_1 = 0 \quad \dots (i)$$

and $A_2x + B_2y + C_2 = 0 \quad \dots (ii)$

From (i) and (ii), we can form an equation

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0, \quad \dots (iii)$$

where k is an arbitrary constant, called *parameter*.

Clearly, for any value of k , the equation (iii) is of first degree in x and y . Hence, it represents a family of lines.

Putting some value of k , we get a particular member of this family.

SOLVED EXAMPLES

EXAMPLE 1 Find the equation of the line drawn through the point of intersection of the lines $4x - 3y + 7 = 0$ and $2x + 3y + 5 = 0$ and passing through the point $(-4, 5)$.

SOLUTION The equation of any line through the point of intersection of the given lines is of the form

$$(4x - 3y + 7) + k(2x + 3y + 5) = 0$$

$$\Rightarrow (4 + 2k)x + (3k - 3)y + (5k + 7) = 0 \quad \dots (i)$$

If it passes through the point $(-4, 5)$, we have

$$(4 + 2k)(-4) + (3k - 3) \cdot 5 + (5k + 7) = 0$$

$$\Rightarrow -16 - 8k + 15k - 15 + 5k + 7 = 0 \Rightarrow 12k = 24 \Rightarrow k = 2.$$

Substituting $k = 2$ in (i), we get

$$8x + 3y + 17 = 0, \text{ which is the required equation.}$$

EXAMPLE 2 Find the equation of the line through the intersection of lines $3x + 4y = 7$ and $x - y + 2 = 0$ and whose slope is 5.

SOLUTION The given lines are $3x + 4y - 7 = 0$ and $x - y + 2 = 0$.

The equation of any line through the point of intersection of the given lines is of the form.

$$(3x + 4y - 7) + k(x - y + 2) = 0 \quad \dots (i)$$

$$\Rightarrow (3 + k)x + (4 - k)y + (2k - 7) = 0$$

$$\Rightarrow (4 - k)y = -(3 + k)x + (7 - 2k)$$

$$\Rightarrow y = \frac{-(3 + k)}{4 - k}x + \frac{(7 - 2k)}{(4 - k)}$$

$$\Rightarrow y = \frac{(k + 3)}{(k - 4)}x + \frac{(7 - 2k)}{(4 - 2k)}.$$



Slope of this line is $\frac{(k+3)}{(k-4)}$.

$$\therefore \frac{k+3}{k-4} = 5 \Rightarrow k+3 = 5k-20 \Rightarrow 4k = 23 \Rightarrow \frac{23}{4}.$$

Substituting $k = \frac{23}{4}$ in (i), we get

$$\begin{aligned} (3x + 4y - 7) + \frac{23}{4}(x - y + 2) &= 0 \\ \Rightarrow 4(3x + 4y - 7) + 23(x - y + 2) &= 0 \\ \Rightarrow 35x - 7y + 18 &= 0, \text{ which is the required equation.} \end{aligned}$$

EXAMPLE 3 Find the equation of the line through the intersection of lines $x + 2y - 3 = 0$ and $4x - y + 7 = 0$ and which is parallel to the line $5x + 4y - 20 = 0$.

SOLUTION $5x + 4y - 20 = 0 \Rightarrow y = \frac{-5}{4}x + 5$

$$\therefore \text{ slope of the given line} = \frac{-5}{4}$$

$$\text{and slope of the required line} = \frac{-5}{4}.$$

Now, the equation of any line through the intersection of the given lines is of the form

$$(x + 2y - 3) + k(4x - y + 7) = 0 \quad \dots (i)$$

$$\Rightarrow (1 + 4k)x + (2 - k)y + (7k - 3) = 0$$

$$\Rightarrow (2 - k)y = -(1 + 4k)x + (3 - 7k)$$

$$\Rightarrow y = \frac{-(1 + 4k)}{(2 - k)}x + \frac{(3 - 7k)}{(2 - k)}$$

$$\Rightarrow y = \frac{(1 + 4k)}{(k - 2)}x + \frac{(3 - 7k)}{(2 - k)}.$$

$$\text{Slope of this line} = \frac{(1 + 4k)}{(k - 2)}$$

$$\therefore \frac{(1 + 4k)}{(k - 2)} = \frac{-5}{4} \Rightarrow 4 + 16k = -5k + 10$$

$$\Rightarrow 21k = 6 \Rightarrow k = \frac{6}{21} = \frac{2}{7}.$$

Substituting $k = \frac{2}{7}$ in (i), we get

$$\begin{aligned} (x + 2y - 3) + \frac{2}{7}(4x - y + 7) &= 0 \\ \Rightarrow (7x + 14y - 21) + (8x - 2y + 14) &= 0 \\ \Rightarrow 15x + 12y - 7 &= 0, \text{ which is the required equation.} \end{aligned}$$

EXAMPLE 4 Find the equation of the line through the intersection of the lines $3x + y - 9 = 0$ and $4x + 3y - 7 = 0$ and which is perpendicular to the line $5x - 4y + 1 = 0$.



SOLUTION $5x - 4y + 1 = 0 \Rightarrow y = \frac{5}{4}x + \frac{1}{4}$.

Slope of the given line is $\frac{5}{4}$.

Now, the equation of any line through the intersection of the given lines is of the form

$$(3x + y - 9) + k(4x + 3y - 7) = 0 \quad \dots (i)$$

$$\Rightarrow (3 + 4k)x + (1 + 3k)y - (9 + 7k) = 0$$

$$\Rightarrow (1 + 3k)y = -(3 + 4k)x + (9 + 7k)$$

$$\Rightarrow y = -\frac{(3 + 4k)}{(1 + 3k)}x + \frac{(9 + 7k)}{(1 + 3k)} \quad \dots (ii)$$

Let m be the slope of the line perpendicular to the required line.

$$\text{Then, } m \times \frac{5}{4} = -1 \Rightarrow m = -\frac{4}{5}.$$

$$\therefore \text{ we must have } \frac{-(3 + 4k)}{(1 + 3k)} = \frac{-4}{5}$$

$$\Rightarrow 15 + 20k = 4 + 12k \Rightarrow 8k = -11 \Rightarrow k = \frac{-11}{8}.$$

Substituting $k = \frac{-11}{8}$ in (i), we get

$$(3x + y - 9) - \frac{11}{8}(4x + 3y - 7) = 0$$

$$\Rightarrow (24x + 8y - 72) - 44x - 33y + 77 = 0$$

$$\Rightarrow 20x + 25y - 5 = 0 \Rightarrow 4x + 5y - 1 = 0.$$

EXAMPLE 5 Find the equation of the line parallel to the y -axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y - 7 = 0$.

SOLUTION The equation of any line through the point of intersection of the given lines is of the form

$$(x - 7y + 5) + k(3x + y - 7) = 0$$

$$\Rightarrow (1 + 3k)x + (k - 7)y + (5 - 7k) = 0 \quad \dots (i)$$

If the line is parallel to y -axis then coefficient of y should be 0, i.e., $k - 7 = 0$, which gives $k = 7$.

Substituting $k = 7$ in (i), we get: $22x - 44 = 0$

$\therefore x - 2 = 0$, which is the required equation.

EXAMPLE 6 Find the equation of the line through the intersection of the lines $2x + 3y = 4$ and $x - 5y + 7 = 0$ that has its x -intercept equal to -4 .

SOLUTION The given lines are $2x + 3y - 4 = 0$ and $x - 5y + 7 = 0$.

The equation of any line through the intersection of the given lines is of the form

$$(2x + 3y - 4) + k(x - 5y + 7) = 0$$

$$\Rightarrow (2 + k)x + (3 - 5k)y + (7k - 4) = 0 \quad \dots (i)$$



If this line has x -intercept -4 , then the point $(-4, 0)$ lies on (i).

$$\therefore (2 + k)(-4) + (7k - 4) = 0 \Rightarrow -8 - 4k + 7k - 4 = 0$$

$$\Rightarrow 3k = 12 \Rightarrow k = 4.$$

Substituting $k = 4$ in (i), we get

$$6x - 17y + 24 = 0, \text{ which is the required equation.}$$

EXERCISE 20K

1. Find the equation of the line drawn through the point of intersection of the lines $x - 2y + 3 = 0$ and $2x - 3y + 4 = 0$ and passing through the point $(4, -5)$.
2. Find the equation of the line drawn through the point of intersection of the lines $x - y = 7$ and $2x + y = 2$ and passing through the origin.
3. Find the equation of the line drawn through the point of intersection of the lines $x + y = 9$ and $2x - 3y + 7 = 0$ and whose slope is $-\frac{2}{3}$.
4. Find the equation of the line drawn through the point of intersection of the lines $x - y = 1$ and $2x - 3y + 1 = 0$ and which is parallel to the line $3x + 4y = 12$.
5. Find the equation of the line through the intersection of the lines $5x - 3y = 1$ and $2x + 3y = 23$ and which is perpendicular to the line $5x - 3y = 1$.
6. Find the equation of the line through the intersection of the lines $2x - 3y = 0$ and $4x - 5y = 2$ and which is perpendicular to the line $x + 2y + 1 = 0$.
7. Find the equation of the line through the intersection of the lines $x - 7y + 5 = 0$ and $3x + y - 7 = 0$ and which is parallel to x -axis.
8. Find the equation of the line through the intersection of the lines $2x - 3y + 1 = 0$ and $x + y - 2 = 0$ and drawn parallel to y -axis.
9. Find the equation of the line through the intersection of the lines $2x + 3y - 2 = 0$ and $x - 2y + 1 = 0$ and having x -intercept equal to 3.
10. Find the equation of the line passing through the intersection of the lines $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ and which cuts off equal intercepts from the axes.

ANSWERS (EXERCISE 20K)

- | | |
|---------------------------|-----------------------|
| 1. $7x + 3y - 13 = 0$ | 2. $4x + 3y = 0$ |
| 3. $2x + 3y - 23 = 0$ | 4. $3x - 4y + 24 = 0$ |
| 5. $63x + 105y - 781 = 0$ | 6. $2x - y - 4 = 0$ |
| 7. $y = 1$ | 8. $x = 1$ |
| 9. $x + 5y - 3 = 0$ | 10. $23x + 23y = 11$ |



7. When a line is parallel to x -axis, then coefficient of x should be 0.

10. Let the required line be

$$\begin{aligned}(3x - 4y + 1) + k(5x + y - 1) &= 0 \\ \Rightarrow (3 + 5k)x + (k - 4)y + (1 - k) &= 0 \\ \Rightarrow (3 + 5k)x + (k - 4)y &= (k - 1) \\ \Rightarrow \frac{(3 + 5k)}{(k - 1)}x + \frac{(k - 4)}{(k - 1)}y &= 1 \Rightarrow \frac{x}{\frac{(k - 1)}{(3 + 5k)}} + \frac{y}{\frac{(k - 1)}{(k - 4)}} = 1 \\ \therefore \frac{k - 1}{3 + 5k} = \frac{k - 1}{k - 4} &\Rightarrow k - 4 = 3 + 5k \Rightarrow 4k = -7 \Rightarrow k = \frac{-7}{4}.\end{aligned}$$